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POSITIVE SOLUTIONS of OPERATOR EQUATIONS

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Positive Solutions of Operator Equations

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ANNOTATION

The book is devoted to a systematic study of an important aspect of non-linear functional analysis. In the book, methods are developed for the study of equations containing essential non-linearities and, in particular, of equations which can have many solutions. The methods developed in the book have already found various applications to problems of wave theory, problems about the forms of the loss of stability of elastic systems, problems of geometry in the large, theory of periodic solutions of equations of non-linear mechanics, theory of non-linear boundary value problems and others.

The book is intended for students of senior courses, aspirants for the doctorate and scientific workers in the various fields of mathematics and mechanics who find it necessary to solve and investigate non-linear problems.

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FOREWORD

In the present book, methods are developed for the investigation of a number of questions connected with the positive solutions of various problems. These questions arise in many divisions of mathematics. In connection with these, general results, obtained in terms of functional analysis, are illustrated by applications to a number of problems: to the first boundary value problem for quasi-linear elliptic equations, to non-linear integral equations, to non-linear oscillations, to the problem about the points of bifurcation, to the theory of the Monge-Ampère equations and others. The applications are based on special constructions and use the properties of the Green functions for various differential operators. Examples of the applications are given in the book but they are not, of course, exhaustive; at the present time, for example, the methods being studied obtained applications to the problems of wave theory, theory of elasticity and others.

In the book, methods are studied for the proof of the existence of positive solutions. Special criteria are found for the uniqueness of a positive solution. The dependence of the solutions on parameters is studied. The question about the convergence of successive approximations to positive solutions is considered. All these questions require the selection of new classes of operators. The study of these classes of operators in turn establishes new facts of the geometry of cones in Banach spaces.

With regard to the study of this book, a knowledge of the fundamentals of functional analysis covered in a general university course is necessary (for example, the first chapters of the book of L. A. Lyusternik and V. I. Sobelev [1] or the book of G. E. Šilov [1]); all the concepts connected with partially ordered spaces are described in detail in the book. In a number of places, elementary concepts of combinatorial topology are applied.

The most important sections of the book were studied in a series of reports in the seminar on functional analysis at the Voronež State University. I am pleased to note that separate arguments (and sometimes im-

portant results) of my comrades and students I. A. Bakhtin, L. A. Ladyzenskii, A. V. Levin, A. I. Perov, Ya. B. Rutickii, P. E. Sobolevskii, V. Ya. Stečenko are presented in the book (see the bibliographical notes at the end of the book). V. Ya. Stečenko rendered a great help in the writing of several chapters of the book. The last section of the seventh chapter was written in collaboration with I. Ya. Bakel'man.

I am obliged to M. G. Krein for his interest in the theory of positive solutions; a number of his ideas have been used in the book.

Voronež, October 1960
M. A. Krasnosel'skii

Chapter 1

SPACES WITH A CONE

In this chapter, we shall discuss the concept of a cone in a Banach space. A partial ordering relation is defined in a space by means of a cone. Various properties of the partial ordering are defined by the geometrical characteristics of cones. In this connection, several classes of cones are studied in this chapter, and criteria for membership of a cone in one class or another, and so on, are indicated.

1.1. Fundamental definitions

1.1.1. *Cones*

As usual, let E be a real Banach space. A set $K \subset E$ is called a *cone* if the following conditions are satisfied:

(a) the set K is closed;

(b) if $u, v \in K$ then $\alpha u + \beta v \in K$ for all $\alpha, \beta \geqslant 0$;

(c) of each pair of vectors (points) x, $-x$ at least one does not belong to K, provided $x \neq \theta$, where θ is the zero of the space E.

From property (b), it follows, in particular, that a cone K is a convex set. A cone is called *solid* if it contains interior points. A cone is called *reproducing* if every element $x \in E$ can be represented in the form

$$x = u - v \qquad (u, v \in K). \tag{1.1}$$

The elements u and v are not defined uniquely in the representation (1.1). Every solid cone is reproducing; in fact, if v_0 is an interior element of the cone, then the element $u = v_0 + \varrho x$, for a sufficiently small $\varrho > 0$, will belong to the cone; this means that the element x is representable in the form

$$x = \frac{u}{\varrho} - \frac{v_0}{\varrho},$$

that is, in the form (1.1).

The collection K_+ of the non-negative functions in the space C (the

space of the functions which are continuous on a bounded closed set) and the collection L_p (the space of the functions which are pth power summable on a bounded set) are the simplest examples of cones. It is not difficult to see that the cone of non-negative functions in the space C is solid; the function $u_0(t)\equiv 1$ is an interior point of this cone. In the case of the spaces L_p, the cone of the non-negative functions will not be solid, but it will be reproducing, since every function $x(t)\in L_p$ can be represented in the form

$$x(t)=x_+(t)-x_-(t)\,,$$

where

$$x_+(t)=\begin{cases} x(t), & \text{if } x(t)\geqslant 0\,,\\ 0, & \text{if } x(t)<0\,,\end{cases}$$

and

$$x_-(t)=\begin{cases} 0, & \text{if } x(t)\geqslant 0\,,\\ -x(t), & \text{if } x(t)<0\,,\end{cases}$$

where $x_+(t)$, $x_-(t)$ are non-negative and belong to L_p.

Let us now consider the space C of continuous functions on the interval $[a, b]$ and select from it the collection K_1 of non-negative, non-decreasing functions. K_1 will be a cone. This cone is not reproducing since only functions of bounded variation can be represented in the form of a difference of two non-decreasing functions.

1.1.2. *The cone $K(F)$*

We point out one general method of constructing cones in Banach spaces.

Let F be a bounded, closed and convex set, not containing the zero θ of the space E. We shall denote by $K(F)$ the collection of all elements $x\in E$ allowing the representation $x=tz$, where $t\geqslant 0$ and $z\in F$. We shall show that $K(F)$ is a cone.

Let $u_n\in K(F)$ $(n=1, 2, \ldots)$ and $\|u_n-v\|\to 0$, $v\neq\theta$. Every element u_0 is representable in the form $u_n=t_n z_n$, where $z_n\in F$. It is obvious that positive constants m and M exist such that $m\leqslant\|z_n\|\leqslant M$. Therefore, the numbers t_n are bounded below and above by positive constants. We select from the sequence t_n the convergent subsequence t_{n_i}, the limit t_0 of which will be a positive number. Obviously

$$\left\|z_{n_i}-\frac{v}{t_0}\right\|\leqslant\frac{1}{t_0}\,|t_0-t_{n_i}|\,\|z_{n_i}\|+\frac{1}{t_0}\,\|u_{n_i}-v\|\,,$$

from which it follows that $(1/t_0)v \in F$. Hence, $v \in K(F)$. We have shown that $K(F)$ is a closed set.

Let $u, v \in K(F)$, i.e. $u = t_1 z_1$, $v = t_2 z_2$, where $t_1, t_2 \geqslant 0$ and $z_1, z_2 \in F$. Let α and β be arbitrary positive numbers. It is obvious that

$$\alpha u + \beta v = (\alpha t_1 + \beta t_2)\left[\frac{\alpha t_1}{\alpha t_1 + \beta t_2} z_1 + \frac{\beta t_2}{\alpha t_1 + \beta t_2} z_2\right].$$

The element, written in the square brackets, belongs to F. Therefore, condition (b) is satisfied.

We must yet verify condition (c). Let us assume that this condition is not satisfied. Then an element $z_0 \in F$ can be found such that $-t_0 z_0 \in K(F)$ for some $t_0 > 0$. This means that

$$-t_0 z_0 = t_1 z_1,$$

where $t_1 > 0$, $z_1 \in F$. It follows from the last equality that

$$\theta = \frac{t_0}{t_0 + t_1} z_0 + \frac{t_1}{t_0 + t_1} z_1 \in F.$$

We have thus come to a contradiction.

1.1.3. *Partial ordering*

The space E is called *partially ordered* if, for certain pairs of elements $x, y \in E$, the relation $x \leqslant y$ is defined, and if the sign $\leqslant$ possesses the usual properties of the sign $\leqslant$. What we have in mind are the following properties:

(1) it follows from $x \leqslant y$ that $tx \leqslant ty$ for $t \geqslant 0$ and $tx \geqslant ty$ for $t < 0$;
(2) it follows from $x \leqslant y$ and $y \leqslant x$ that $x = y$;
(3) if $x_1 \leqslant y_1$ and $x_2 \leqslant y_2$, then it follows that $x_1 + x_2 \leqslant y_1 + y_2$;
(4) it follows from $x \leqslant y$ and $y \leqslant z$ that $x \leqslant z$.

A partial ordering is introduced in a Banach space E with a cone K in the following manner: write $x \leqslant y$ if $y - x \in K$. Properties (1), (3), and (4) follow from condition (b); property (2) follows from condition (c). (We leave it to the reader to show this.)

One more important property of a partial ordering, which is defined by means of a cone, follows from condition (a):

(5) one can pass to the limit in the inequalities: if $\|x_n - x\| \to 0, \|y_n - y\| \to 0$ and $x_n \leqslant y_n$ $(n = 1, 2, \ldots)$, then $x \leqslant y$.

If K is the cone of non-negative functions in the space C or L_p (or in some other space of functions), then the partial ordering relation has a simple

meaning: $x(t) \leqslant y(t)$, if $x(t) \leqslant y(t)$ for all (or almost all in the case of the space L_p) values of t.

1.1.4. *Auxiliary lemmas*

Let x_0 and v_0 be arbitrary fixed constant elements of the space E, which is partially ordered by means of a cone K. We shall consider the collection Π of elements x of the form

$$x = x_0 + tv_0 \qquad (-\infty < t < \infty). \tag{1.2}$$

If $v_0 \neq \theta$, then all the elements (1.2) cannot belong to the cone K, since in the contrary case the elements $(x_0/|t|) + v_0 \operatorname{sgn} t$ would belong to K for arbitrarily large $|t|$ and, by virtue of the fact that K is closed, the elements v_0 and $-v_0$ would belong to K.

The assertion, proved above, has a simple geometrical interpretation: no straight line (1.2) lies completely in the cone. It follows from the convexity of the cone that the set of those values t, for which the element (1.2) belongs to the cone (if such t's exist), either forms a finite closed interval $[t_0, t_1]$, or one of two half-infinite closed intervals: either $(-\infty, t_0]$ or $[t_0, \infty)$. It is not difficult to see that the case where the interval is $[t_0, \infty)$ occurs only when $v_0 \in K$, and the case where the interval is $(-\infty, t_0]$ occurs only when $-v_0 \in K$. The above line of reasoning leads to the following lemmas:

Lemma 1.1. *Let $u_0 \in K$, $x \in E$. Then it follows from $x \leqslant tu_0$ that $x \leqslant \gamma u_0$ for all $\gamma > t$.*

Lemma 1.2. *Let $u_0 \in K$. For an element $x \in E$, suppose a t_1 can be found such that $x \leqslant t_1 u_0$. Then a smallest t exists for which $x \leqslant tu$.*

Lemma 1.3. *Let $u_0 \in K$. For an element $x \in E$, suppose a t_1 can be found such that $x \geqslant -t_1 u_0$. Then a smallest t exists for which $x \geqslant -tu_0$.*

1.2. Normal cones

1.2.1. *Definitions*

A cone K is called *normal* if a $\delta > 0$ exists such that $\|e_1 + e_2\| \geqslant \delta$ for e_1, $e_2 \in K$ and $\|e_1\| = \|e_2\| = 1$. The cones of non-negative functions, considered in the preceding examples, are normal.*

The important concept of a *u_0-norm* will be utilized for the study of normal

cones, and which, moreover, will also be applied later in other problems. Let u_0 be a fixed non-zero element of the cone K. An element $x \in E$ will be called *u_0-measurable* if non-negative numbers t_1 and t_2 can be found such that

$$-t_1 u_0 \leqslant x \leqslant t_2 u_0 . \tag{1.3}$$

We shall denote the greatest lower bound of the numbers t_1 and t_2 by $\alpha(x)$ and $\beta(x)$ respectively. By virtue of Lemmas 1.2 and 1.3,

$$-\alpha(x) u_0 \leqslant x \leqslant \beta(x) u_0 , \tag{1.4}$$

where inequalities (1.3) are satisfied for all $t_1 > \alpha(x)$, $t_2 > \beta(x)$.

We shall denote the collection of all u_0-measurable elements $x \in E$ by E_{u_0}. It is obvious that E_{u_0} is a linear set; it can be converted to a normed space if we set

$$\|x\|_{u_0} = \max \{\alpha(x), \beta(x)\} . \tag{1.5}$$

Verification of the norm axioms does not present any difficulty, and so we leave this task to the reader. In the sequel, we shall call the norm (1.5) a *u_0-norm*.

Let the element x^* satisfy the inequalities

$$\alpha_0 u_0 \leqslant x^* \leqslant \beta_0 u_0 \qquad (\alpha_0, \beta_0 > 0) .$$

Then every u_0-measurable element $x \in E$ will be x^*-measurable and, conversely, every x^*-measurable element will be u_0-measurable. Thus, in the case under consideration, E_{u_0} and E_{x^*} consist of the same elements. It is not difficult to see that then the u_0-norm and the x^*-norm are equivalent.

1.2.2. *Criterion for the normality of a cone*

Theorem 1.1. *A necessary and sufficient condition for a cone K to be normal is that the inequality*

$$\|x\|_E \leqslant M \|x\|_y \, \|y\|_E \qquad (x \in E_y, \; y \in K, \; y \neq \theta) \tag{1.6}$$

is satisfied, where the constant M is independent of both x and y.

Proof. Let the cone K be normal. We shall assume that it is impossible to

---

* The cone of non-negative functions in the space C_1 of functions $x(t)$ which are continuously differentiable on $[a, b]$ with the norm

$$\|x(t)\|_{C_1} = \max_t |x(t)| + \max_t |x'(t)|$$

does not possess the property of normality.

find a constant M for which condition (1.6) is satisfied. Then sequences $x_n \in E_{y_n}$, $y_n \in K$ can be found such that

$$\|x_n\|_E \geqslant n\|x_n\|y_n\|y_n\|_E \qquad (n=1, 2, \ldots),$$

from which it follows that

$$-\frac{y_n}{n\|y_n\|_E} \leqslant \frac{x_n}{\|x_n\|_E} \leqslant \frac{y_n}{n\|y_n\|_E}.$$

Therefore, the elements

$$g_n = \frac{x_n}{\|x_n\|_E} + \frac{y_n}{n\|y_n\|_E}, \qquad h_n = -\frac{x_n}{\|x_n\|_E} + \frac{y_n}{n\|y_n\|_E}$$

belong to K. It follows from the normality of the cone that

$$\left\| \frac{g_n}{\|g_n\|_E} + \frac{h_n}{\|h_n\|_E} \right\|_E \geqslant \delta \qquad (n=1, 2, \ldots),$$

where δ is a positive number. But a direct calculation shows that

$$\frac{g_n}{\|g_n\|_E} + \frac{h_n}{\|h_n\|_E} = \frac{2y_n}{n\|y_n\|_E\|g_n\|_E} + \frac{\|g_n\|_E - \|h_n\|_E}{\|g_n\|_E\|h_n\|_E} h_n,$$

and since

$$1 - \frac{1}{n} \leqslant \|g_n\|_E, \quad \|h_n\|_E \leqslant 1 + \frac{1}{n}, \quad |\,\|g_n\|_E - \|h_n\|_E| \leqslant \frac{2}{n},$$

we have

$$\left\| \frac{g_n}{\|g_n\|_E} + \frac{h_n}{\|h_n\|_E} \right\|_E \leqslant \frac{4}{n-1},$$

from which it follows that

$$\lim_{n\to\infty} \left\| \frac{g_n}{\|g_n\|_E} + \frac{h_n}{\|h_n\|_E} \right\|_E = 0.$$

We have thus arrived at a contradiction.

The converse of the theorem is proved quite simply. Let condition (1.6) be satisfied and let $x, y \in K$, $\|x\| = \|y\| = 1$. Then

$$\|x\|_E \leqslant M\|x\|_{x+y}\|x+y\|_E. \tag{1.7}$$

But $\theta \leqslant x \leqslant x+y$; that is, $\|x\|_{x+y} \leqslant 1$. Therefore

$$\|x\|_E \leqslant M\|x+y\|_E, \tag{1.8}$$

from which it follows that

$$\|x+y\|_E \geqslant \frac{1}{M}.$$

This completes the proof of the theorem.

In particular, inequality (1.6) means that the original norm in the space E is always subordinate to every u_0-norm, provided the cone K is normal. If the cone K does not possess the property of normality, then an element $u_0 \in K$ can be found such that $\|x\|_E$ is not subordinate to the u_0-norm. In fact, if K does not possess the property of normality, then sequences

$$x_n, y_n \in K, \quad \|x_n\| = \|y_n\| = 1 \qquad (n = 1, 2, \ldots)$$

can be constructed such that

$$\|x_n + y_n\| \leqslant \frac{1}{n^3} \qquad (n = 1, 2, \ldots);$$

then, setting

$$u_0 = \sum_{n=1}^{\infty} n(x_n + y_n),$$

we obtain the obvious inequalities

$$\theta \leqslant x_n \leqslant \frac{1}{n} u_0 \qquad (n = 1, 2, \ldots),$$

from which it follows that

$$\|x_n\|_{u_0} \leqslant \frac{1}{n} = \frac{1}{n} \|x_n\|_E.$$

The example constructed does not contradict the fact that in the special choice of the element u_0, the usual norm will be subordinate to the u_0-norm, in spite of the fact that the cone K does not possess the property of normality.

The observations discussed in this subsection are of a simple geometrical nature. Let T be a centrally symmetric convex set in a linear space E, where T does not contain completely a straight line $tx (-\infty < t < \infty)$. We shall denote by $E(T)$ the collection of elements $x \in E$ such that $(x/\lambda) \in T$ for some $\lambda > 0$. We shall denote by $\|x\|_T$ the infimum of those $\lambda > 0$ for which $(x/\lambda) \in T$. It can be verified that $\|x\|_T$ satisfies all the norm axioms. Therefore, $E(T)$ can be regarded as a normed space.

Let us assume that two centrally symmetric convex sets T_1 and T_2 are considered, where $T_1 \subset T_2$. It follows from the definition given above that $E(T_1) \subset E(T_2)$, where

$$\|x\|_{T_2} \leqslant \|x\|_{T_1} \quad (x \in E(T_1)).$$

If $T_2 = kT_1$ (in the sense that T_2 consists of elements of the form kx, where $x \in T_1$), then it is obvious that

$$E(T_2)=E(T_1)$$

and

$$\|x\|_{T_1}=k\|x\|_{T_2}=k\|x\|_{kT_1}\,.$$

Now let F_1 and F_2 be arbitrary convex centrally symmetric sets. From the above line of reasoning, it follows that the inclusion $F_1\subset F_2$ implies the subordination of the norm $\|x\|_{F_2}$ to the norm $\|x\|_{F_1}$. The converse statement is also valid.

1.2.3. *Monotonic and semi-monotonic norms*

We shall call a norm in the space E with cone K *semi-monotonic* if $x\leqslant y$ implies that $\|x\|\leqslant N\|y\|$ for arbitrary elements $x, y\in K$, where N is a numerical constant.

We shall call the norm *monotonic* if $\theta\leqslant x\leqslant y$ implies the inequality $\|x\|\leqslant\|y\|$. It is not difficult to see that the norm in spaces C and L_p is monotonic with respect to the partial ordering generated by the cone of non-negative functions.

Theorem 1.2. *A necessary and sufficient condition for the normality of the cone is that the norm be semi-monotonic.*

Proof. Let the cone K be normal. Then, for arbitrary $x, y\in K$, $x\leqslant y$, we have, by virtue of Theorem 1.1, that

$$\|x\|_E\leqslant M\|x\|_y\ \|y\|_E\leqslant M\|y\|_E\,,$$

since $\|x\|_y\leqslant 1$.

Conversely, let the norm be semi-monotonic. Then, $\|x\|\leqslant N\|x+y\|$ for $x, y\in K$ and hence $\|x+y\|\geqslant 1/N$ for $x, y\in K$, $\|x\|=\|y\|=1$.

This completes the proof of the theorem.

As we have already mentioned, the cones of non-negative functions in the spaces C and L_p are normal, which can be verified directly. The normality of these cones also follows from Theorem 1.2.

The cones $K(F)$ were considered above (section 1.2, subsection 1.2.2). These cones are also normal. (We shall obtain this fact further on as an obvious corollary to the theorems on regular cones.)

We note yet that Theorems 1.1 and 1.2 remain valid if the cones are considered in normed spaces which do not possess the property of completeness.

1.3. The space E_{u_0}

1.3.1. *Examples*

In subsection 1.2.1 of the previous section, u_0-norms and the space E_{u_0} were considered. We shall give some examples.

Let E be the space L_p of functions which are pth power summable $(1 \leqslant p < \infty)$ on a bounded set Ω. Let K be the cone of non-negative functions. We choose as u_0 the function which is identically equal to 1. Then all functions which are bounded in absolute value will be u_0-measurable; that is, E_{u_0} coincides with L_∞:

$$\|x\|_{u_0} = \|x(t)\|_{L_\infty} = \operatorname{vrai} \max_{t \in \Omega} |x(t)| \,. \tag{1.9}$$

Since, in the example being considered, the cone K is normal, we have by virtue of Theorem 1.1 that

$$\|x\|_E \leqslant M \|x\|_{x_0} \,;$$

moreover, the last inequality also follows from (1.9):

$$\|x(t)\|_{L_p} \leqslant (\operatorname{mes} \Omega)^{1/p} \|x(t)\|_{L_\infty} \,.$$

Again, let $E = L_p$ and let K be the cone of non-negative functions. Let $u_0 = u_0(t) = |u_0(t)| \in L_{p_1}$, where $p_1 \geqslant p$. It is obvious that such functions $x(t)$ for which

$$|x(t)| \leqslant a u_0(t) \tag{1.10}$$

will be u_0-measurable, where the u_0-norm will be the smallest a for which inequality (1.10) is satisfied. For the functions $x(t)$, which satisfy inequality (1.10), the evident inequality

$$\|x(t)\|_{L_{p_1}} = \left\{ \int_\Omega |x(t)|^{p_1} dt \right\}^{1/p_1} \leqslant a \|u_0(t)\|_{L_{p_1}}$$

also is satisfied, from which it follows that

$$\|x(t)\|_{L_{p_1}} \leqslant \|u_0(t)\|_{L_{p_1}} \|x(t)\|_{u_0} \qquad (x(t) \in E_{u_0}) \,.$$

In this way, from convergence with respect to the u_0-norm follows, in the case considered, the convergence in L_{p_1} and a fortiori in L_p itself; the last conclusion could also have been obtained on the basis of Theorem 1.1.

As our last example, we shall consider the space C of continuous functions on a bounded closed set Ω with the cone K of non-negative functions. If $u_0(t) \equiv 1$, then the u_0-norm will be the usual norm of the space C. If, for

example, $\Omega=[0, 1]$ and $u_0(t)=t(1-t)$, then the u_0-measurable functions will be the functions for which

$$|x(t)|\leqslant at(1-t)\,,$$

where the smallest a is the u_0-norm of the function $x(t)$; the sequence of the functions $x_n(t)$ will, obviously, converge with respect to the u_0-norm if the functions $x_n(t)/[t(1-t)]$ converge in L_∞. Convergence with respect to the u_0-norm implies, by virtue of Theorem 1.1, convergence in the space C; this fact is also easily obtained directly since

$$\|x(t)\|_C\leqslant\|x(t)\|_{u_0}\max_{t\in[0,1]}t(1-t)=\tfrac{1}{4}\|x(t)\|_{u_0}\,. \tag{1.11}$$

1.3.2. *Completeness of the space* E_{u_0}

Theorem 1.3. *Let the cone K be normal. Then the space E_{u_0} is complete with respect to the u_0-norm.*

Proof. Let $x_n\in E_{u_0}(n=1, 2, \ldots)$ be a fundamental sequence with respect to the u_0-norm.

By virtue of (1.6), this sequence will also be fundamental with respect to the usual norm of the space E and, consequently, an element x^* can be found such that $\|x_n-x^*\|_E\to0$. The fact that the sequence x_n is fundamental with respect to the u_0-norm means that for every n we have

$$-\varepsilon_n u_0\leqslant x_{n+m}-x_n\leqslant\varepsilon_n u_0\qquad(m=1, 2, \ldots)\,,$$

where $\varepsilon_n\to0$. Passing to the limit as $m\to\infty$, we obtain the relations

$$-\varepsilon_n u_0\leqslant x^*-x_n\leqslant\varepsilon_n u_0\,.$$

Consequently, $x^*\in E_{u_0}$ and $\|x^*-x_n\|_{u_0}\leqslant\varepsilon_n$.

This completes the proof of the theorem.

Let us assume that the cone K is solid and that u_0 is an interior element of the cone. We shall then denote by ϱ the radius of the sphere with center at u_0, lying completely in K. It is obvious that for an arbitrary $x\in E$, the inequality

$$-u_0\leqslant\varrho\,\frac{x}{\|x\|_E}\leqslant u_0$$

will be satisfied, from which it follows that

$$-\frac{\|x\|_E}{\varrho}u_0\leqslant x\leqslant\frac{\|x\|_E}{\varrho}u_0$$

and

$$\|x\|_{u_0}\leqslant\frac{1}{\varrho}\|x\|_E\,.$$

Thus, in the case under consideration, the u_0-norm is subordinate to the usual norm.
It is not difficult to see that the converse statement is also valid: if $E_{u_0}=E$ and the u_0-norm is subordinate to the usual norm, then the cone K is solid and u_0 is an interior element of K.

1.3.3. *The cone* K_{u_0}
We introduce the notation $K_{u_0}=K\cap E_{u_0}$. It is obvious that K_{u_0} satisfies conditions (b) and (c) of the definition of a cone.

Lemma 1.4. *Let* $x^*\in E$ *and* $x^*\bar{\in}K$. *Then* x^* *is not a limit element for* K_{u_0} *with respect to the* u_0*-norm.*

Proof. Assuming the contrary, a sequence $x_n\in K$ can be found such that $-u_0/n\leqslant x_n-x^*\leqslant u_0/n$. Then $x^*+(u_0/n)\in K$ and $x^*\in K$ by virtue of the fact that K is closed.
We have thus arrived at a contradiction and this completes the proof of the lemma.

This lemma means that K_{u_0} is a set which is closed with respect to the u_0-norm in E_{u_0}. From this and from Theorem 1.3, it follows that K_{u_0} is a cone in the Banach space E_{u_0} if K is a normal cone in E.

Lemma 1.5. *The* u_0*-norm is monotonic on* E_{u_0}.

Proof. Let $\theta\leqslant x\leqslant y$. Then

$$\theta\leqslant x\leqslant\|y\|_{u_0}u_0\,,$$

from which it follows that $\|x\|_{u_1}\leqslant\|y\|_{u_0}$. This completes the proof of the lemma.
We emphasize that in the conditions of the lemmas no restrictions were placed on the cone K.
From the lemmas, proved above, it follows that K_{u_0} is a normal cone in the space E_{u_0} if K is normal. In this connection, K_{u_0} is a solid cone. A more general statement occurs in the following subsection.

1.3.4. *The cone* $\bar{K}_{u_0}$
If the cone K does not possess the property of normality, then E_{u_0} might not possess the property of completeness with respect to the u_0-norm. In this case, we shall denote by $\bar{K}_{u_0}$ the closure of K_{u_0} in the space $\bar{E}_{u_0}$,

which is obtained from E_{u_0} by completion with respect to the u_0-norm.* By virtue of Lemma 1.4,

$$\bar{K}_{u_0} \cap E_{u_0} = K_{u_0} .$$

Lemma 1.6. *$\bar{K}_{u_0}$ is a cone.*

Proof. Condition (b) (see p. 17) in the definition of a cone follows from the validity of this same property for K_{u_0} by means of passage to the limit. We shall show that property (c) is satisfied. Assuming the contrary, sequences $x_n, y_n \in K_{u_0} (n = 1, 2, \ldots)$ exist such that

$$\|x_n - x^*\|_{u_0} \to 0, \quad \|y_n + x^*\|_{u_0} \to 0 ,$$

where $x^* \in \bar{E}_{u_0}$, $\|x^*\|_{u_0} > 0$. It follows from this that

$$\|x_n + y_n\|_{u_0} \to 0 ,$$

and, by virtue of Lemma 1.5,

$$\|x_n\|_{u_0} \to 0, \quad \|y_n\|_{u_0} \to 0 .$$

But then $x^* = \theta$ and we have thus arrived at a contradiction.
This completes the proof of the lemma.

In the space $\bar{E}_{u_0}$ we shall define a partial ordering by means of the cone $\bar{K}_{u_0}$. We recall that $\bar{E}_{u_0} = E_{u_0}$ and $\bar{K}_{u_0} = K_{u_0}$ if the cone K is normal.
By virtue of Lemma 1.4, the partial ordering, defined on E_{u_0} by means of the cone $\bar{K}_{u_0}$, coincides with the partial ordering which was introduced by means of the cone K.

Theorem 1.4. *The cone $\bar{K}_{u_0}$ is normal and solid in the space E_{u_0}; the u_0-norm is monotonic with respect to the partial ordering defined by the cone $\bar{K}_{u_0}$.*

Proof. If $\|x - u_0\|_{u_0} \leqslant 1$ and $x \in E$, then $-u_0 \leqslant x - u_0 \leqslant u_0$, from which it follows that $x \geqslant \theta$, that is, $x \in K_{u_0}$. Passing to the limit, we come to the conclusion that the sphere of unit radius with respect to the u_0-norm with center at the point u_0 lies completely in $\bar{K}_{u_0}$. The fact that the cone $\bar{K}_{u_0}$ is solid is thus proved.
We shall now prove that the u_0-norm is monotonic. From this, by virtue

* If $E = C_1$ (see footnote on p. 21) and $u_0(t) = 1$, then E_{u_0} does not possess the property of completeness with respect to the u_0-norm which coincides with the norm in the space C. Completion to $\bar{E}_{u_0}$ can be carried out by adjoining all continuous functions to E_{u_0}.

of Theorem 1.2, the normality of the cone $\overline{K}_{u_0}$ will follow.
Let $x \leqslant y$, where x and y are arbitrary elements from $\overline{K}_{u_0}$. The elements $x+(u_0/n)$ and $y+(u_0/n)$ will be interior points of the cone $\overline{K}_{u_0}$. Therefore, for each fixed n, elements x_n and y_n can be formed from K_{u_0} such that

$$\left\| x + \frac{u_0}{n} - x_n \right\|_{u_0} < \frac{1}{n}, \qquad \left\| y + \frac{u_0}{n} - y_n \right\|_{u_0} < \frac{1}{n}. \tag{1.12}$$

It obviously follows from the last inequalities that

$$-\left(x + \frac{u_0}{n} - x_n\right) \leqslant \frac{u_0}{n}, \qquad y + \frac{u_0}{n} - y_n \leqslant \frac{u_0}{n}, \tag{1.13}$$

where, in the last relations, the partial ordering is defined by the cone $\overline{K}_{u_0}$. From (1.13) and from $x \leqslant y$ it follows that

$$\begin{aligned} x_n &= x - \left(x + \frac{u_0}{n} - x_n\right) + \frac{u_0}{n} \leqslant x + \frac{2u_0}{n} \leqslant y + \frac{2u_0}{n} \\ &= y_n + \left(y + \frac{u_0}{n} - y_n\right) + \frac{u_0}{n} \leqslant y_n + \frac{2u_0}{n}, \end{aligned}$$

where x_n and $y_n + (2u_0/n)$ are elements of K_{u_0}, by virtue of which

$$y_n + \frac{2u_0}{n} - x_n \in K_{u_0}.$$

This implies the inequality

$$\theta \leqslant x_n \leqslant \left(\|y_n\|_{u_0} + \frac{2}{n}\right) u_0;$$

that is,

$$\|x_n\|_{u_0} \leqslant \|y_n\|_{u_0} + \frac{2}{n}.$$

But then, by virtue of (1.12),

$$\|x\|_{u_0} \leqslant \|x_n\|_{u_0} + \left\| x + \frac{u_0}{n} - x_n \right\|_{u_0} + \frac{1}{n} \leqslant \|y_n\|_{u_0} + \frac{4}{n} \leqslant \|y\|_{u_0} + \frac{6}{n}. \tag{1.14}$$

Passing to the limit in (1.14) as $n \to \infty$, we obtain the inequality

$$\|x\|_{u_0} \leqslant \|y\|_{u_0}.$$

This completes the proof of the theorem.

1.4. Linear positive functionals

1.4.1. *Positive functionals*

A functional $f(x)$, defined on K (or on a subset of K), is called *positive* if $f(x) \geqslant 0$ for $x \in K$. The functional $f(x)$ is called *monotonic* if $f(x_1) \leqslant f(x_2)$ for $x_1 \leqslant x_2$ $(x_1, x_2 \in K)$.

The norm of an element is the simplest example of a monotonic functional in the case of spaces with a monotonic norm. By virtue of Theorem 1.4, the u_0-norm is a monotonic functional on K_{u_0}.

The most important class of positive functionals are linear positive functionals, that is, functionals $f(x)$ defined on all the space E, such that

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

and

$$f(x) \geqslant 0 \qquad (x \in K).$$

Every linear positive functional is evidently monotonic.

We recall some general theorems on linear functionals in Banach spaces. Let $f(x)$ be a linear continuous functional defined on E. We shall denote by $G\{f=c\}$ the hyperplane consisting of $x \in E$ such that $f(x)=c$. The hyperplane $G\{f=c\}$ decomposes the entire space into two closed half-spaces $E\{f \geqslant c\}$ and $E\{f \leqslant c\}$, on which the functional $f(x)$ takes values not smaller than c and not greater than c respectively. In this connection, the hyperplane $G\{f=c\}$ belongs to both half-spaces. The formula

$$|f(x)-c| = \|f\| \varrho(x, G\{f=c\}), \tag{1.15}$$

which is analogous to the usual formula of analytical geometry, by which the distance from a point to a plane is defined, holds. Let T_1 and T_2 be two convex sets. We shall assume that T_1 is solid, i.e. T_1 has interior points, in which connection, none of the interior points of the set T_1 belongs to T_2. In this case, it turns out that a hyperplane G can be constructed such that the convex sets T_1 and T_2 will lie in different closed half-spaces, into which the hyperplane G decomposes the space E. The last statement can be formulated analytically as follows: a linear continuous functional $f(x)$ exists such that $f(x) \geqslant c$ for $x \in T_1$ and $f(x) \leqslant c$ for $x \in T_2$. By virtue of (1.15), the values of the functional $f(x)$ will be different from c if x is an interior point of T_1.

Let K be a cone in E. If $x_0 \overline{\in} K$, then, by virtue of the fact that K is closed, a sphere $\|x-x_0\| < \varrho$ can be found which does not have points in common with K. By virtue of the statement, formulated above, a linear continuous

functional $f(x)$ can be found such that $f(x) \geqslant c$ for $x \in K$, where c is a constant. But then $f(nx) \geqslant c$ for arbitrary n, that is,

$$f(x) \geqslant \frac{c}{n},$$

from which it follows that

$$f(x) \geqslant 0.$$

Thus, *linear positive functionals exist.*

It turns out that a stronger statement can be made: for every $x_0 \in K (x \neq \theta)$ a positive linear continuous function $f(x)$ can be found such that $f(x_0) > 0$.

If the space E is separable, then a linear continuous functional $f(x)$ can be constructed such that $f(x_0) > 0$ for all $x \in K$, $x \neq \theta$.

We note one more fact. If the cone K is solid, then the positiveness of an additive homogeneous functional implies its continuity.★

1.4.2. *Uniformly positive functionals*

We shall call the positive linear functional $f(x)$ *uniformly positive* if a positive number a exists such that

$$f(x) \geqslant a\|x\| \qquad (x \in K). \tag{1.16}$$

In the space L of functions which are summable on some set Ω of functions with the cone K of non-negative functions, the functional

$$f(x) = \int_{\Omega} x(t)\,dt$$

will be uniformly positive, since $f(x) = \|x\|$ for $x \in K$.

In the spaces L_p, where $p > 1$, there are no uniformly positive (on the cone K of non-negative functions) linear functionals. Let us assume the contrary, that for some functional $f(x)$, condition (1.16) is satisfied. We divide the set Ω into n subsets, $\Omega_1, \Omega_2, \ldots, \Omega_n$ of equal measure, and we define the functions $x_1(t), x_2(t), \ldots, x_n(t)$ by the equality

$$x_i(t) = \begin{cases} n^{1/p}, & \text{if} \quad t \in \Omega_i \qquad (i = 1, 2, \ldots, n), \\ 0, & \text{if} \quad t \bar{\in} \Omega_i \qquad (i = 1, 2, \ldots, n). \end{cases}$$

It is obvious that

★ The reader can find the proofs of the statements presented in this subsection in the article of M. G. Krein and M. A. Rutman [1]. The last statement of the subsection allows the following generalization: the positiveness of an additive homogeneous functional implies its continuity if the cone is reproducing (see the footnote on p. 64).

$$\|x_i(t)\|_{L_p} = (\text{mes } \Omega)^{1/p} \qquad (i = 1, 2, \ldots, n).$$

Then, by virtue of (1.16),

$$f(x_i) \geqslant a(\text{mes } \Omega)^{1/p} \qquad (i = 1, 2, \ldots, n)$$

and

$$f(x^*) = f\left(\frac{1}{n^{1/p}} \sum_{i=1}^{n} x_i\right) \geqslant an^{1-(1/p)}(\text{mes } \Omega)^{1/p},$$

where $x^*(t) \equiv 1$. The last inequality contradicts the finiteness of $f(x^*)$. **In the space C of continuous functions, there are no uniformly positive linear functionals (with respect to the cone of non-negative functions) either. It is not difficult to give a direct proof of this statement. Later, it will be obtained as a simple corollary to general theorems.**

1.4.3. *Cones which allow plastering*

We shall say that cone K *allows the plastering* K_1 if a cone K_1 can be found such that every non-zero element $x_0 \in K$ is an interior point of the cone K_1 and, furthermore, it lies in the cone K_1 together with the spherical neighborhood of radius $b\|x_0\|$, where b does not depend on the element x_0. We shall sometimes say that the cone K_1 is *plastered by the cone* K.

Theorem 1.5. *A necessary and sufficient condition that a uniformly positive linear functional can be defined uniformly on a cone K is that the cone K allows plastering.*

Proof. In one direction, the statement is almost obvious. In fact, if K allows the plastering K_1 then every positive functional $f(x)$ on K_1 will be uniformly positive on K, because, by virtue of (1.15),

$$f(x_0) \geqslant b\|x_0\| \, \|f\| \qquad (x_0 \in K).$$

In the other direction, the theorem is proved in a somewhat more complicated manner.

Let $f(x)$ be a uniformly positive linear functional on K and let

$$f(x) \geqslant a\|x\| \qquad (x \in K). \tag{1.17}$$

We shall denote by N the set of points x for which $f(x) = 1$. The set $M = K \cap N$ will be convex and, by virtue of (1.17), bounded. Let x^* be a fixed element in M. We shall denote by F the collection of $x \in N$ such that $\|x - x^*\| \leqslant 2\varrho$, where ϱ is a fixed positive number so large that

$$\|x - x_1\| \leqslant \varrho \qquad (x_1 \in M, \; x \in N)$$

implies the membership of x in the set F. Further, we construct a cone $K_1 = K(F)$ (see p. 18) of elements u which allow the representation $u = tx$, where $t \geqslant 0$ and $x \in F$. We shall show that K_1 is the plastering of the cone K. In fact, let $x_0 \in K$ and

$$\|h\| \leqslant \frac{\varrho a^2}{2\|f\| + \varrho a \|f\|} \|x_0\| . \tag{1.18}$$

We shall show that $x_0 + h \in K_1$. For this it is sufficient to show that

$$\left\| \frac{x_0 + h}{f(x_0 + h)} - \frac{x_0}{f(x_0)} \right\| \leqslant \varrho , \tag{1.19}$$

since the elements $(x_0 + h)/[f(x_0 + h)]$ and $x_0/[f(x_0)]$ evidently belong to N, where the second one belongs to M. The inequality (1.19) is equivalent to the inequality

$$\|f(x_0)h - f(h)x_0\| \leqslant \varrho f(x_0)[f(x_0) + f(h)] ,$$

which follows from (1.17) and (1.18) since

$$\|f(x_0)h - f(h)x_0\| \leqslant 2\|f\| \, \|x_0\| \, \|h\| \leqslant \frac{2\varrho a^2 \|x_0\|^2}{2 + \varrho a}$$

and

$$\varrho f(x_0)[f(x_0) + f(h)] \geqslant \varrho a \|x_0\| (a\|x_0\| - \|f\| \, \|h\|) \geqslant$$

$$\geqslant \varrho a \|x_0\|^2 \left(a - \frac{\varrho a^2}{2 + \varrho a} \right) = \frac{2\varrho a^2 \|x_0\|^2}{2 + \varrho a} .$$

This completes the proof of the theorem.

We shall say that the cone K is *locally compact* if every bounded set $F \subset K$ is compact. It is obvious that every locally compact cone lies in some separable subspace $E_1 \subset E$. Therefore, a positive linear functional $f(x)$ can be defined, satisfying the condition

$$f(x) > 0 \qquad (x \in K, \; x \neq \theta) .$$

This functional is uniformly positive since $f(x) \geqslant a\|x\|$, where

$$a = \min_{x \in K, \, \|x\| = 1} f(x) > 0 .$$

It then follows from Theorem 1.5 that *every locally compact cone allows plastering.*

Let us note one more property of cones which allow plastering.

We recall that a sequence $x_n \in E (n = 1, 2, \ldots)$ is called *weakly convergent in itself* if the numerical sequence $f(x_n)$ converges for an arbitrary linear functional $f(x) \in E^*$. The sequence x_n is *weakly convergent* to the element x^* if $f(x_n) \to f(x^*)$ for all $f \in E^*$.

Lemma 1.7. *Let the cone K allow plastering. Let the sequence of normed elements x_n, $\|x_n\|=1$ $(n=1, 2, \ldots)$, belonging to the cone K, converge weakly to some element x^*. Then*

$$x^* \neq \theta .$$

Proof. Let $f(x)$ be a uniformly positive linear functional on K. Then

$$f(x^*) = \lim_{n\to\infty} f(x_n) \geqslant a \lim_{n\to\infty} \|x_n\| = a ,$$

from which it follows that $\|x^*\| \geqslant a/\|f\|$.

This completes the proof of the lemma.

The statement of the lemma is false for normal cones. For example, let K be the cone of non-negative functions in the space $L_p = L_p(\Omega)(1<p<\infty)$. We consider a sequence of the sets $\Omega_n \subset \Omega$ such that

$$\text{mes } \Omega_n = \frac{1}{n} \qquad (n=1, 2, \ldots) ,$$

and we set

$$x_n(t) = \begin{cases} n^{1/p}, & \text{if } \ t\in\Omega_n , \\ 0 , & \text{if } \ t\bar{\in}\Omega_n . \end{cases}$$

It is obvious that $\|x_n(t)\| = 1$ $(n=1, 2, \ldots)$. At the same time, the sequence $x_n(t)$ weakly converges to zero, since for an arbitrary function

$$v(t)\in L_q \left(\frac{1}{p} + \frac{1}{q} = 1\right) ;$$

it follows, by virtue of the Hölder inequality, that

$$|(x_n, v)| = \left|\int_\Omega x_n(t)v(t)\,dt\right| = \left|\int_{\Omega n} n^{1/p} v(t)dt\right| \leqslant \left\{\int_{\Omega n} |v(t)|^q\,dt\right\}^{1/q}$$

and from the absolute continuity of the integral

$$\int_\Omega |v(t)|^q\,dt$$

that

$$\lim_{n\to\infty} (x_n, v) = 0 .$$

1.4.4. *Example of a cone which allows plastering*

Let F be a bounded closed convex set not containing the zero of the space E. We shall show that the cone $K(F)$ allows plastering. For the proof, we shall construct a hyperplane $f(x)=c$, separating the convex set F and the sphere $\|x\| \leqslant \varrho$ which is disjoint from F, where ϱ is a fixed number.

Without loss of generality, we can assume that $c>0$. From the definition of the cone $K(F)$, it follows that, for each non-zero $x\in K(F)$, a positive number t can be found such that $tx/\|x\|\in F$, where the boundedness of F implies that the numbers t are uniformly bounded by some positive number γ. Then, it follows from the inequality $f(tx/\|x\|)>c$ that

$$f(x)\geqslant \frac{c}{\gamma}\|x\| \qquad (x\in K(F)) .$$

Hence, the functional $f(x)$ is uniformly positive. By virtue of Theorem 1.5, the cone $K(F)$ allows plastering.

One particular class of cones $K(F)$ will be used later in chapter 6, section 6.4. Let a subspace E_0 be selected in E, the defect of which is equal to 1. This means that there exists an element $h_0\in E(\|h_0\|=1)$ such that every element $x\in E$ allows a unique representation of the form

$$x=x'+\xi(x)h_0 \qquad (x'\in E_0) ,$$

where $\xi(x)$ is a linear functional. We shall denote by P the linear operator of projection on E_0 which is defined by the equality

$$Px=x'=x-\xi(x)h_0 \qquad (x\in E) .$$

We denote by $K(h_0,\varrho)$ the collection of elements $x\in E$ for which the inequalities

$$\xi(x)\geqslant 0 , \quad \|Px\|\leqslant\varrho\xi(x) \tag{1.20}$$

are satisfied.

It is not difficult to see that $K(h_0,\varrho)$ is a cone $K(F)$, where F is the collection of $x\in E$ such that $\xi(x)=1$ and $\|Px\|\leqslant\varrho$. Therefore, every cone $K(h_0,\varrho)(0<\varrho<\infty)$ allows plastering.

Collections of convex, non-negative functions give interesting examples of cones which allow plastering.

For example, let Ω be a bounded convex domain in a two-dimensional plane. We shall denote by K the collection of convex functions on Ω, which vanish on the boundary Γ of the domain Ω; this collection is a cone. The cone K allows plastering, since the value of the function at some fixed interior point of the domain Ω is a uniformly positive linear functional on K.

1.5. Regular cones

1.5.1. *Definitions*

In this section, we shall consider the question of what additional conditions guarantee the existence of a limit of a monotonic sequence $x_n (n=1, 2, \ldots)$ of elements of the space E, which is partially ordered by means of the cone K. We shall restrict ourselves to the case when the sequence x_n does not decrease:

$$x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n \leqslant \ldots \tag{1.21}$$

We shall call the sequence (1.21) *bounded* if there exists an element y such that

$$x_n \leqslant y \qquad (n=1, 2, \ldots). \tag{1.22}$$

It must straight away be emphasized that the boundedness of the sequence (1.21) does not imply its convergence. Thus, for example, the sequence of functions

$$x_n(t)=1-t^n \qquad (n=1, 2, \ldots) \tag{1.23}$$

satisfies condition (1.21) if it is considered in the space C of continuous functions on [0, 1], which is partially ordered by means of the cone of non-negative functions. This sequence is evidently bounded. At the same time, the sequence (1.23) does not have a limit in C.

However, spaces exist in which (1.21) and (1.22) imply convergence in norm. For example, if E is the space L_p, partially ordered by means of the cone of non-negative functions, then, for every non-decreasing bounded sequence of the functions $x_n(t)$ (bounded in the sense of (1.22)), the function

$$x^*(t)=\sup_n x_n(t) \qquad (t \in \Omega) \tag{1.24}$$

will also belong to $L_p(\Omega)$. The sequence $x_n(t)$ converges in measure to $x^*(t)$ and, by virtue of the Lebesgue theorem about passage to the limit under the integral sign,

$$\lim_{n\to\infty} \|x_n - x^*\|_{L_p} = \lim_{n\to\infty} \left\{\int_\Omega |x_n(t)-x^*(t)|^p \, dt\right\}^{1/p} = 0,$$

that is, the sequence $x_n(t)$ has a limit in L_p.

The above examples motivate the selection of a class of spaces in which each bounded monotonic sequence has a limit. In the sequel, we shall call such spaces *regularly partially ordered*. A cone which generates a regular partial ordering will be called *regular*.

1.5.2. *The relation between the regularity and normality of a cone*
Theorem 1.6. *Every regular cone is normal.*

Proof. Let us assume that the cone K does not possess the property of normality. Then, sequences $x_n \in K$, $y_n \in K$ can be found such that

$$\|x_n\| = \|y_n\| = 1, \quad \|x_n + y_n\| < \frac{1}{n^2} \qquad (n = 1, 2, \ldots). \tag{1.25}$$

The series

$$(x_1 + y_1) + (x_2 + y_2) + \ldots + (x_n + y_n) + \ldots$$

converges by virtue of (1.25); we shall denote its sum by u. We shall define the sequence z_n by means of the formula

$$z_n = x_1 + x_2 + \ldots + x_n \qquad (n = 1, 2, \ldots).$$

It is obvious that

$$\|z_{n+1} - z_n\| = 1 \qquad (n = 1, 2, \ldots).$$

The sequence z_n is monotonic and bounded since

$$z_n \leqslant (x_1 + y_1) + (x_2 + y_2) + \ldots + (x_n + y_n) \leqslant u.$$

However, by virtue of (1.25), this sequence does not converge in norm in E. Thus, the cone K does not possess the property of regularity.
This completes the proof of the theorem.
The converse of Theorem 1.6 is false. A normal cone might not possess the property of regularity (cf. the cone of non-negative functions in the space C).

1.5.3. *Fully regular cone*
A sequence x_n is called *bounded in norm* if

$$\|x_n\| \leqslant M \qquad (n = 1, 2, \ldots),$$

where M is some number. We shall call the cone K *fully regular* if every monotonic norm-bounded sequence converges (in norm) to some limit. This definition is justified by the following theorem:

Theorem 1.7. *Every fully regular cone is regular.*

Proof. We shall first show that every fully regular cone is normal. Let us assume the contrary. Then, sequences $e_n, g_n \in K$ $(n = 1, 2, \ldots)$ can be found such that

$$\|e_n\| = \|g_n\| = 1, \quad \|e_n + g_n\| < \frac{1}{2^n}.$$

The sequence

$$h_k = \begin{cases} e_1+g_1+e_2+g_2+ \dots +e_n+g_n, & \text{if } k=2n, \\ e_1+g_1+e_2+g_2+ \dots +e_n+g_n+e_{n+1}, & \text{if } k=2n+1, \end{cases}$$

is evidently monotonic, that is,

$$h_1 \leqslant h_2 \leqslant \dots \leqslant h_n \leqslant \dots,$$

and bounded in norm: for $k=2n$,

$$\|h_k\| \leqslant \|e_1+g_1\| + \|e_2+g_2\| + \dots + \|e_n+g_n\| \leqslant \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n} < 1$$

and for $k=2n+1$,

$$\|h_k\| \leqslant \|e_1+g_1\| + \dots + \|e_n+g_n\| + \|e_{n+1}\| < 2.$$

At the same time, the sequence h_k does not converge in norm because $\|h_{k+1}-h_k\|=1$. Thus, the cone K does not possess the property of full regularity. We have thus arrived at a contradiction.
Now let the monotonic sequence

$$x_1 \leqslant x_2 \leqslant \dots \leqslant x_n \leqslant \dots$$

be bounded in the sense of the partial ordering:

$$x_n \leqslant y \qquad (n=1, 2, \dots).$$

The sequence x_n-x_1 will also be non-decreasing and bounded by the element $y-x_1$. By Theorem 1.1, the sequence of norms of the elements x_n-x_1 will be bounded, since the normality of cone K is already proved. From the full regularity of the cone, it follows that the sequence x_n-x_1 converges in norm. It follows from this that the sequence x_n also converges in norm.
This completes the proof of the theorem.

It will be shown in the following section that in some spaces the cone of non-negative functions is regular but does not possess the property of full regularity. In the present and following subsections, sufficient conditions will be given, such that the regularity of a cone implies its full regularity when these conditions are satisfied.

Theorem 1.8. *If a regular cone K is solid, then it is fully regular.*

Proof. Obviously, it suffices to show that every sequence which is bounded in norm is bounded in the sense of a partial ordering.
Let u be an interior element of the cone K, occurring in K together with a sphere of radius ϱ. The fact that $u_0-\varrho(x/\|x\|)\in K$ for all $x\in E(x\neq\theta)$,

implies the inequality

$$x \leqslant \frac{\|x\|}{\varrho} u_0 .$$

This completes the proof of the theorem.

1.5.4. *The conjugate cone*

The linear normed space E^* of all linear continuous functionals on E is called the *conjugate space* to E.

The positive linear functionals on the cone K also form a cone if K is reproducing.* The cone of positive functionals is called the *conjugate* cone and is denoted by K^*.

The following theorem of M. G. Kreĭn holds: *the cone K is normal if, and only if, the cone K^* is reproducing.*

The space E is called *weakly complete* if every weakly convergent in-itself sequence has a weak limit. The spaces $L_p (1<p<\infty)$ are weakly complete.

Theorem 1.9. *Let the space E be weakly complete. Then every regular cone K is fully regular.*

Proof. As in the proof of the previous theorem, it suffices to show that a non-decreasing sequence, bounded in norm, is also bounded in the sense of a partial ordering.

Let

$$x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n \leqslant \ldots$$

and

$$\|x_n\| \leqslant M \qquad (n=1, 2, \ldots) .$$

Then for every positive linear functional $f(x)$, the sequence $f(x_n)$ converges. From the theorem of M. G. Kreĭn, formulated above, it follows that an arbitrary functional $l(x)$ from E^* is representable in the form of a difference

$$l(x)=f_1(x)-f_2(x)$$

of two positive functionals. Therefore, for an arbitrary linear functional $l(x)$, the numerical sequence $l(x_n)$ converges. It follows from the weak completeness of the space E that the sequence x_n converges weakly to some element $x^* \in E$.

---

* It is sufficient that the closure of the linear hull of the cone K coincides with E. This assumption guarantees property (c) in the definition of a cone. Properties (a) and (b) are obvious.

Again, let $f(x)$ be a positive functional. It is obvious that

$$f(x_n) \leqslant f(x^*) \qquad (n=1, 2, \ldots).$$

Therefore,

$$x_n \leqslant x^* \qquad (n=1, 2, \ldots).$$

This completes the proof of the theorem.

1.5.5. *Additional remarks*

If the cone K does not possess the property of regularity, then the boundedness of a monotonic sequence is not sufficient for its convergence; in this case, it is necessary to impose additional restrictions.

In the sequel, we shall frequently make use of the fact, for instance, that *compact, monotonic sequences have a limit.* For the proof of this statement, it is necessary to note, in the first place, that the limits of all convergent subsequences of a monotonic sequence coincide and, in the second place, that a compact sequence converges if the limits of all its subsequences are equal.

In conclusion, let us note that a regular cone K in the space E can lose its property to be regular in the transition to the space E_{u_0}. In other words, a bounded monotonic sequence of u_0-measurable elements might not converge with respect to the u_0-norm. We shall give an example of such a situation. Let E be the space $L_p(\Omega)$, $\Omega=[0, 1]$, with the cone K of non-negative functions, and let $u_0(t)=1$. As already noted, the cone K possesses the property of regularity. The sequence of functions (1.23) belongs to $E_{u_0}=L_\infty$, and is monotonic and bounded, but it does not converge with respect to the u_0-norm.

1.6. Criteria for the regularity of a cone

1.6.1. *Strictly increasing functionals*

A positive (not necessarily linear) functional $f(x)$ will be called *strictly increasing* if, for arbitrary $x_n \in K$ $(n=1, 2, \ldots)$,

$$\|x_n\| \geqslant \varepsilon_0 > 0 \qquad (n=1, 2, \ldots)$$

implies that

$$\lim_{n\to\infty} f(x_1+x_2+\ \ldots\ +x_n)=\infty\,. \tag{1.26}$$

Linear uniformly positive functionals (if they exist) can serve as the simplest examples of strictly increasing functionals, that is, linear functionals $f(x)$ such that

$$f(x) \geqslant a\|x\| \qquad (x \in K).$$

In the space $L_p (1 \leqslant p < \infty)$, the pth power of the norm of the element

$$f(x) = \int_\Omega |x(t)|^p \, dt$$

is a strictly increasing functional. For the proof, it suffices to use the inequality

$$(\alpha+\beta)^p \geqslant \alpha^p + \beta^p \qquad (\alpha, \beta \geqslant 0)^\star, \tag{1.27}$$

by virtue of which

$$f(x+y) \geqslant f(x) + f(y)$$

for arbitrary non-negative functions $x(t), y(t) \in L_p$.

We recall that a functional $f(x)$ is called monotonic if $x \leqslant y$ implies that $f(x) \leqslant f(y)$.

1.6.2. *Basic criteria*

Theorem 1.10. *Let the strictly increasing functional $f(x)$, which is bounded on every sphere, be defined on the cone K. Then the cone K is fully regular.*

Proof. Assuming the contrary, we can find a sequence $x_n \in K$ which is not convergent in norm such that

$$x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n \leqslant \ldots \tag{1.28}$$

and

$$\|x_n\| \leqslant M \qquad (n = 1, 2, \ldots). \tag{1.29}$$

We shall assume that $\|x_1\| \geqslant \varepsilon_0$, $\|x_{n+1} - x_n\| \geqslant \varepsilon_0 > 0$ $(n = 1, 2, \ldots)$, since otherwise one can go over to a subsequence possessing this property. Then

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f\left[x_1 + \sum_{i=1}^{n-1} (x_{i+1} - x_i)\right] = \infty \ ;$$

but this contradicts the boundedness of $f(x)$ on the sphere $\|x\| \leqslant M$. This completes the proof of the theorem.

It follows from this theorem, in particular, that the cones of non-negative functions in the spaces $L_p (1 \leqslant p \leqslant \infty)$ are fully regular.

$\star$ For the proof of (1.27), it is sufficient to integrate the obvious inequality $p(1+t)^{p-1} \geqslant pt^{p-1} (t \geqslant 0)$ between the limits from 0 to β/α.

Theorem 1.11. *Let a monotonic, strictly increasing functional be defined on the cone K. Then the cone K is regular.*

Proof. Assuming the contrary, we can find a sequence $x_n \in K$ such that the inequalities (1.28) and the inequalities

$$x_n \leqslant z \qquad (n=1, 2, \ldots) \tag{1.30}$$

are satisfied.

As with the proof of Theorem 1.10, we arrive at the equality

$$\lim_{n\to\infty} f(x_n) = \infty ,$$

which contradicts the inequalities

$$f(x_n) \leqslant f(z) \qquad (n=1, 2, \ldots) .$$

This completes the proof of the theorem.

1.6.3. *Examples of regular, but not fully regular cones**

Let $M(u)$ be an N-function, that is, a convex function which is representable in the form

$$M(u) = \int_0^{|u|} p(t)\,dt ,$$

where the function $p(t)$, which is positive for $t>0$, is continuous on the right, monotonic and such that

$$\lim_{t\to+0} p(t) = 0, \quad \lim_{t\to+\infty} p(t) = \infty .$$

We denote by $N(v)$ the conjugate $M(u)$-function defined by the equality

$$N(v) = \int_0^{|v|} q(s)\,ds ,$$

where

$$q(s) = \sup_{p(t)\leqslant s} t .$$

The functions

$$\frac{1}{p}|u|^p \quad (1<p<\infty) , \qquad |u|^p(|\ln|u||+1) \qquad (1<p<\infty) ,$$

$$e^{|u|}-|u|-1 , \quad e^{u^2}-1$$

can serve as examples of N-functions.

* For a detailed discussion of those facts of the theory of Orlicz spaces which are utilized in the present subsection, see the book by M. A. Krasnosel'skii and Ya. B. Rutickii [1].

The so-called Δ_2-condition of Young plays an important role in the theory of convex functions. One says that $M(u)$ satisfies this condition if a k and an a exist such that

$$M(2u) \leqslant kM(u) + a$$

for all $u \geqslant 0$.

From the examples given above, the first two N-functions satisfy the Δ_2-condition but the last two do not.

We shall denote by $L_M^* = L_M^*(\Omega)$ the collection of all functions $x(t)$ which are measurable on Ω, for which

$$\int_\Omega x(t)y(t)\,dt < \infty$$

for an arbitrary $y(t)$ such that

$$\varrho(y; N) = \int_\Omega N[y(t)]\,dt < \infty\,.$$

The set L_M^* becomes a complete Banach space if we set

$$\|x(t)\| = \sup_{\varrho(y;N)\leqslant 1} \int_\Omega x(t)y(t)\,dt\,.$$

The space L_M^* is called an *Orlicz space*.

The space L_M^* contains the function $|x(t)|$ whenever it contains the function $x(t)$. From this it follows that the collection of non-negative functions forms a reproducing cone in L_M^*. This cone is normal since $\theta \leqslant x(t) \leqslant y(t)$ (that is, the fulfillment of the inequality $0 \leqslant x(t) \leqslant y(t)$ for almost all $t \in \Omega$) implies that

$$\|x(t)\| \leqslant \|y(t)\|\,.$$

01. *If $M(u)$ satisfies the Δ_2-condition, then the cone K of non-negative functions in L_M^* is fully regular.*

Proof. If $M(u)$ satisfies the Δ_2-condition, then L_M^* consists of those functions $x(t)$ for which

$$F(x) = \int_\Omega M[x(t)]\,dt < \infty\,.$$

We shall show that the function $F(x)$ is strictly increasing on K and is bounded on every sphere; then statement 01 will follow from the general theorems which were proved previously.

The inequality

$$M(u_1 + u_2) \geqslant M(u_1) + M(u_2)$$

is satisfied for every N-function when $u_1, u_2 \geqslant 0$. Therefore, for arbitrary non-negative functions $h_1(t), \ldots, h_n(t)$, the inequality

$$F(h_1 + \ldots + h_n) = \int_\Omega M[h_1(t) + \ldots + h_n(t)]\,dt \geqslant$$
$$\geqslant \int_\Omega M[h_1(t)]\,dt + \ldots + \int_\Omega M[h_n(t)]\,dt =$$
$$= F(h_1) + \ldots + F(h_n)$$

is satisfied.

In the case when the Δ_2-condition is satisfied, the relation

$$\lim_{n\to\infty} F(h_n) = \lim_{n\to\infty} \int_\Omega M[h_n(t)]\,dt = 0$$

implies that

$$\lim_{n\to\infty} \|h_n(t)\| = 0 .$$

Therefore, to each $\varepsilon_0 > 0$ there can be assigned a $\delta_0 > 0$ such that $\|h(t)\| \geqslant \varepsilon_0$ implies the inequality

$$F(h) = \int_\Omega M[h(t)]\,dt \geqslant \delta_0 .$$

Let us assume that $h_n(t) \in K$ and $\|h_n(t)\| \geqslant \varepsilon_0$ $(n = 1, 2, \ldots)$. Then

$$F(h_1 + \ldots + h_n) \geqslant F(h_1) + \ldots + F(h_n) \geqslant n\delta_0 .$$

Thus, $F(x)$ is a strictly increasing functional.

It remains to show that the functional $F(x)$ is bounded on every sphere $\|x(t)\| \leqslant R$. Let $R < 2^m$. It then follows from the Δ_2-condition that

$$M(u) \leqslant kM\left(\frac{u}{2}\right) + a \leqslant k^2 M\left(\frac{u}{4}\right) + a(1+k) \leqslant \ldots \leqslant$$
$$\leqslant k^m M\left(\frac{u}{2^m}\right) + a(1 + k + \ldots + k^{m-1}) \leqslant k_1 M\left(\frac{u}{R}\right) + a_1 ,$$

where $k_1 = k^m$, $a_1 = a(1 + k + \ldots + k^{m-1})$. Consequently,

$$F(x) = \int_\Omega M[x(t)]\,dt \leqslant k_1 \int_\Omega M\left[\frac{x(t)}{R}\right] dt + a_1 \operatorname{mes} \Omega .$$

If $\|x(t)\| \leqslant 1$, then the inequality

$$\int_\Omega M[u(t)]\,dt \leqslant \|u(t)\| \leqslant 1$$

holds. Thus, we obtain the estimate

$$F(x) \leqslant k_1 \left\| \frac{x(t)}{R} \right\| + a_1 \operatorname{mes} \Omega \leqslant k_1 + a_1 \operatorname{mes} \Omega \quad (\|x(t)\| \leqslant R) .$$

This completes the proof of statement 01.

In particular, if $M(u)=|u|^p/p\,(1<p<\infty)$, then $L_M{}^*=L_p$; therefore, the full regularity of the cone of non-negative functions in L_p follows from statement 01. This fact was established earlier by the same method.

02. *If $M(u)$ does not satisfy the Δ_2-condition, then the cone of non-negative functions in $L_M{}^*$ does not even possess the property of regularity.*

Proof. Suppose $M(u)$ does not satisfy the Δ_2-condition. Then the set of functions which are bounded on Ω is nowhere dense in $L_M{}^*$. We shall denote by E_M the closure in $L_M{}^*$ of the set of bounded functions.
We shall denote by $z(t)$ some non-negative function from $L_M{}^*$, which does not belong to $E_M{}^*$. We form the sequence

$$x_n(t) = \begin{cases} z(t), & \text{if } 0 \leqslant z(t) \leqslant n, \\ n, & \text{if } n \leqslant z(t). \end{cases}$$

This sequence is evidently monotonic and is bounded by the function $z(t)$:

$$x_1(t) \leqslant x_2(t) \leqslant \ldots \leqslant x_n(t) \leqslant \ldots \leqslant z(t).$$

At the same time, the sequence $x_n(t)$ does not converge in norm.
In fact, if the sequence $x_n(t)$ did converge to some function $x^*(t)$, then it would also converge in measure to $x^*(t)$; that is, the sequence would converge in norm to $z(t)$. At the same time,

$$\lim_{n\to\infty} \|x_n(t)-z(t)\| = \inf_{y(t)\in E_M} \|z(t)-y(t)\| > 0.$$

Hence the cone K does not possess the property of regularity.
This completes the proof of statement 02.

03. *If $M(u)$ does not satisfy the Δ_2-condition, then the cone K of non-negative functions in the space E_M is regular, but not fully regular.*

Proof. The fact that the cone K is not fully regular in the space E_M was, in essence, established in the proof of the previous statement. The sequence of functions $x_n(t)$ considered there is monotonic and bounded in norm since

$$\|x_n(t)\| \leqslant \|z(t)\| \qquad (n=1, 2, \ldots);$$

at the same time, this sequence does not converge in norm.
It remains to prove the regularity of the cone K.
By definition, the function $x(t)\in L_M{}^*$ has an absolutely continuous norm if to each $\varepsilon>0$, there corresponds a $\delta>0$ such that for every set $\Omega_1\subset\Omega$,

mes $\Omega_1 < \delta$ implies the inequality

$$\|x(t)\varkappa(t; \Omega_1)\| < \varepsilon ,$$

where $\varkappa(t; \Omega_1)$ denotes the characteristic function of the set Ω_1. By definition, the sequence of the functions $x_n(t)$ have absolutely equi-continuous norms if to each $\varepsilon > 0$ there corresponds a $\delta > 0$ such that the inequality

$$\|x_n(t)\varkappa(t; \Omega_1)\| < \varepsilon$$

holds for each $\Omega_1 \subset \Omega$ and simultaneously for all n provided mes $\Omega_1 < \delta$.
A necessary and sufficient condition for a sequence of functions $x_n(t)$ to converge in norm in E_M is that it converges in measure and has absolutely equi-continuous norms.
We shall consider in E_M a sequence of functions $x_n(t)$ which is monotonic and bounded, in the sense of a partial ordering, by some function $z(t) \in E_M$:

$$x_1(t) \leqslant x_2(t) \leqslant \ldots \leqslant x_n(t) \leqslant \ldots \leqslant z(t) .$$

Without loss of generality, we can consider all the functions $x_n(t)$ to be non-negative.
It follows from the fact that the $z(t)$ belongs to the space E_M that it has an absolutely continuous norm. It follows from the obvious inequality

$$\|x_n(t)\varkappa(t; \Omega_1)\| \leqslant \|z(t)\varkappa(t; \Omega_1)\| \qquad (n = 1, 2, \ldots)$$

that the functions $x_n(t)$ have absolutely equi-continuous norms. From the monotonicity of the sequence $x_n(t)$ it follows that it converges in measure to some function $x^*(t) \leqslant z(t)$. Consequently, $x^*(t) \in E_M$ and $x_n(t)$ converges in norm to $x^*(t)$.
Statement 03 is thus completely proved.

The last statement is of fundamental interest to us. It means that not every regular cone is fully regular.
The collection K of all sequences $x = \{\xi_1, \ldots, \xi_n, \ldots\}$ with non-negative coordinates in the space (s) of sequences which converge to zero with the norm

$$\|x\| = \max_i |\xi_i| .$$

is a simple example of a regular, but not fully regular, cone. This cone is not fully regular since the sequence

$$x^{(k)} = \{\xi_1^{(k)}, \ldots, \xi_n^{(k)}, \ldots\} ,$$

where $\xi_i^{(k)} = 1$ for $i \leqslant k$ and $\xi_i^{(k)} = 0$ for $i > k$, in the space (s), which is monotonic and bounded in norm, does not converge strongly. The regularity of the cone K is obvious.

1.6.4. *About cones which allow plastering*

By virtue of Theorem 1.10, a cone K is fully regular if a uniformly positive linear functional can be defined on it. Therefore, Theorem 1.5 implies the next theorem.

Theorem 1.12. *If a cone K allows plastering, then it is fully regular.*

In particular, the cones $K(F)$ are fully regular.

The cone of non-negative functions in the space C does not possess the property of regularity; therefore, a uniformly positive linear functional cannot be defined on it.

Lemma 1.8. *If a cone K is solid, then the cone K^* allows plastering.*

Proof. Let u_0 be an interior element of the cone K. By virtue of (1.15), the inequality

$$f(u_0) \geqslant \varrho \|f\| \tag{1.31}$$

is satisfied for every $f \in K^+$, where ϱ is the radius of the sphere with center at u_0, belonging entirely to K. The inequality (1.31) means that the linear functional $F(f)$, defined on K^* by the equality

$$F(f) = f(u_0),$$

is uniformly positive. It remains to apply Theorem 1.5. This completes the proof of the lemma.

It follows from this lemma and Theorem 1.12 that the cone K^* is fully regular if K is solid.

We now introduce the concept of the cone $K_{u_0,\varrho}$. Let u_0 be a fixed element from K; $\varrho \geqslant 1$. We shall denote by $K_{u_0,\varrho}$ the collection of elements $x \in K$ such that

$$au_0 \leqslant x \leqslant \varrho a u_0 \tag{1.32}$$

for some $a \geqslant 0$.

We shall show that $K_{u_0,\varrho}$ is a cone. Let $x_1, x_2 \in K_{u_0,\varrho}$; then

$$a_1 u_0 \leqslant x_1 \leqslant \varrho a_1 u_0, \qquad a_2 u_0 \leqslant x_2 \leqslant \varrho a_2 u_0$$

and, for arbitrary non-negative α and β,

$$(\alpha a_1 + \beta a_2) u_0 \leqslant \alpha x_1 + \beta x_2 \leqslant \varrho(\alpha a_1 + \beta a_2) u_0 ;$$

that is, $\alpha x_1 + \beta x_2 \in K_{u_0,\varrho}$. If $x \in K_{u_0,\varrho}$ and $x \neq \theta$, then $-x \notin K_{u_0,\varrho}$ since $K_{u_0,\varrho}$ is a subset of the cone K. It remains to prove that $K_{u_0,\varrho}$ is closed; this follows from the fact that in the relations

$$a_n u_0 \leqslant x_n \leqslant \varrho a_n u_0 \qquad (n=1, ,2, \ldots)$$

one can pass to the limit on some subsequence of indices if the sequence x_n converges strongly.

Theorem 1.13. *If the cone K is normal, then the cone $K_{u_0,\varrho}$ allows plastering.*

Proof. We shall define a linear positive functional $f(x)$ on the cone K which satisfies the condition

$$f(u_0)=1 .$$

We shall show that this functional is uniformly positive on the cone $K_{u_0,\varrho}$. With this, the theorem will be proved.
Since the cone K is normal, the norm is semi-monotonic: $\|x\| \leqslant L\|y\|$ if $x \leqslant y$ and $x, y \in K$.
It follows from the inequality

$$au_0 \leqslant x \leqslant \varrho a u_0 \tag{1.33}$$

that

$$a\|u_0\| \leqslant L\|x\| \leqslant \varrho L^2 a \|u_0\| ,$$

from which we have

$$\frac{1}{\varrho L\|u_0\|}\|x\| \leqslant a \leqslant \frac{L}{\|u_0\|}\|x\| . \tag{1.34}$$

Applying the functional $f(x)$ to (1.33), we obtain

$$a \leqslant f(x) \leqslant \varrho a \qquad (x \in K_{u_0,\varrho})$$

and, by virtue of (1.34),

$$f(x) \geqslant \frac{1}{\varrho L\|u_0\|}\|x\| \qquad (x \in K_{u_0,\varrho}) .$$

The last inequality means that the functional $f(x)$ is uniformly positive. This completes the proof of the theorem.

Now, if we use Theorem 1.12, then we come to the following conclusion: if K is normal, then the cone $K_{u_0,\varrho}$ is fully regular.
We shall give examples of cones $K_{u_0,\varrho}$.
If, in the space C of functions which are continuous on Ω, we consider the collections of positive functions $x(t)$ which satisfy the condition

$$\max_{t\in\Omega} x(t) \leqslant \varrho \min_{t\in\Omega} x(t) , \tag{1.35}$$

then it forms a cone $K_{u_0,\varrho}$, where K is the cone of all non-negative functions, and $u_0(t) \equiv 1$. Analogously, in the spaces L_p, the collection of fun-

tions $x(t)$ for which

$$\operatorname*{vrai\,max}_{t\in\Omega} x(t) \leqslant \varrho \operatorname*{vrai\,min}_{t\in\Omega} x(t) \tag{1.36}$$

forms a cone $K_{u_0,\varrho}$, where $u_0(t)\equiv 1$.

The cones $K_{u_0,\varrho}$ are closely connected with the cone K_{u_0} which was introduced when we considered the u_0-norm (section 1.2). It is obvious that

$$K_{u_0} = \bigcup_{\varrho>1} K_{u_0,\varrho} .$$

1.7. Minihedral cones

1.7.1. *Definition*

The element z is called the least upper bound (*supremum*) of the pair of elements x and y if $x\leqslant z$, $y\leqslant z$ and if it follows from the inequalities $x\leqslant z_1$, $y\leqslant z_1$ that $z\leqslant z_1$. The greatest lower bound (*infimum*) of two elements is defined analogously.

If every pair of elements has a supremum, then the cone K is called *minihedral.*

It is not difficult to see that the minihedrality of a cone is equivalent to the fact that, for arbitrary $x, y\in E$, an element z can be found such that $K_x\cap K_y=K_z$; here, K_u denotes the collection of elements of the form $u+x$ $(x\in K)$. In this connection, $z=\sup\{x, y\}$, that is

$$K_x\cap K_y=K_{\sup\{x,y\}} . \tag{1.37}$$

The cones of non-negative functions in the spaces C and L_p are minihedral.

If the cone K is minihedral, then the least upper bound exists for every finite collection of elements $x_1, x_2, \ldots, x_n$, that is, there exists an element z such that

$$x_i\leqslant z \qquad (i=1, 2, \ldots, n) ,$$

where the inequalities

$$x_i\leqslant z_1 \qquad (i=1, 2, \ldots, n)$$

imply the relation $z\leqslant z_1$. The least upper bound can be defined by the equality

$$\sup\{x_1, x_2, \ldots, x_n\}=\sup\{x_1, \sup\{x_2, \ldots, x_n\}\} .$$

We also note the formula

$$K_{x_1}\cap K_{x_2}\cap \ldots \cap K_{x_n}=K_{\sup\{x_1, \ldots, x_n\}} , \tag{1.38}$$

which is analogous to formula (1.37).
Under certain assumptions, a minihedral cone can be considered as the cone of non-negative functions in the space of functions which are continuous on some bicompactum. We shall not give a precise formulation, and especially not a proof, of this fundamental theorem of M. G. Krein and S. G. Krein [1]. We note only that the following result of a negative nature follows from it: a solid minihedral cone in an infinite-dimensional space cannot be regular.

1.7.2. *Existence of the least upper bound for a denumerable set of elements* Theorem 1.14. *Let the cone K be regular and minihedral. Then every bounded sequence $x_n (n=1, 2, \ldots)$ has a least upper bound.*

Proof. Let

$$x_n \leqslant z \qquad (n=1, 2, \ldots). \tag{1.39}$$

Then it is obvious that

$$y_n = \sup\{x_1, x_2, \ldots, x_n\} \leqslant z \qquad (n=1, 2, \ldots). \tag{1.40}$$

Since the sequence y_n increases monotonically and is bounded, it converges in norm to some limit z_0.
Passing to the limit in the inequality $y_n \leqslant y_{n+k}$ as $k \to \infty$, we obtain $y_n \leqslant z_0$ $(n=1, 2, \ldots)$, from which it follows that

$$x_n \leqslant z_0 \qquad (n=1, 2, \ldots).$$

Let us assume that the relations (1.39) are satisfied. Then, the relations (1.40) are satisfied, in which, upon passing to the limit as $n \to \infty$, we obtain the inequality $z_0 \leqslant z$. This means that z_0 is the least upper bound for the sequence x_n.
This completes the proof of the theorem.

The statement of Theorem 1.14 can be interpreted geometrically: if the sequence x_n is bounded, then

$$\bigcap_{n=1}^{\infty} K_{x_n} = K_{\sup\{x_1, x_2, \ldots, x_n\}}. \tag{1.41}$$

The assumption about the regularity of the cone in Theorem 1.14 is essential. For example, the cone of non-negative functions in the space C is minihedral, but does not possess the property of regularity; bounded sequences exist in C which do not have a supremum (for example, sequence (1.23)).

Let the conditions of Theorem 1.14 be satisfied. Then, for every non-decreasing bounded sequence x_n the equality

$$\lim_{n\to\infty} \|x_n - x^*\| = 0\,, \tag{1.42}$$

where

$$x^* = \sup\{x_1, x_2, \ldots\}\,, \tag{1.43}$$

is valid.

For the proof, it suffices to observe that x_n converges in norm to some element z_0. Then, on the one hand, it follows from $x_n \leqslant z_0$ that $x^* \leqslant z_0$, and, on the other, $x_n \leqslant x^*$ implies that $z_0 \leqslant x^*$.

1.7.3. *Strongly minihedral cones*

A non-linear functional $f(x)$, defined on a cone K, is called *strictly monotonic* if $x \leqslant y (x \neq y)$ implies that $f(x) < f(y)$.

Uniformly positive linear functionals can serve as examples of strictly monotonic functionals. As we have already noted, a strictly monotonic linear continuous functional can be constructed for every cone in a separable space.

Theorem 1.15. *Let K be a regular and minihedral cone. Let a strictly monotonic functional $f(x)$ (i.e. $x \leqslant y$ and $x \neq y$ imply that $f(x) < f(y)$) be defined on K, which possesses the property that $\|x_n - x^*\| \to 0$ implies that $f(x_n) \to f(x^*)$ for every non-decreasing bounded sequence x_n.*
Then every bounded set has a least upper bound.

Proof. Let M be a bounded set: $x \leqslant z_0$ for all $x \in M$. Without loss of generality, it can be assumed that the set M contains the point θ (otherwise, we would consider the set of elements of the form $x - x_0$, where x ranges over all of M and x_0 is a fixed element of M; the supremum of the new set, after the addition of x_0 to it, will give sup M).

We consider all elements y of the form

$$y = \sup\{\theta, x_1, \ldots, x_k\}\,, \tag{1.44}$$

where $x_1, \ldots, x_k$ are arbitrary elements from M. It is obvious that all the elements (1.44) satisfy the inequality $y \leqslant z_0$. Therefore, $f(y) \leqslant f(z_0)$.

We select a sequence $y_n (n = 1, 2, \ldots)$ of elements of the form (1.44) so that the equality

$$\lim_{n\to\infty} f(y_n) = \sup f(y) \tag{1.45}$$

holds, where the supremum is taken with respect to all elements of the form (1.44). In this connection, we shall assume that $y_1 \leqslant y_2 \leqslant \ldots \leqslant y_n \leqslant \ldots$ The sequence y_n is bounded and, therefore, it converges in norm to some element y^*. We shall prove that $y^* = \sup M$.

If z is an arbitrary upper bound of the space M, then, in particular, $y_n \leqslant z (n = 1, 2, \ldots)$. After passage to the limit, we arrive at the relation $y^* \leqslant z$. It remains to show that y^* is the least upper bound for the set M.

Let $x_0 \in M$. We introduce the notation $y_0 = \sup \{y^*, x_0\}$.

It is necessary to prove that $y_0 = y^*$. Assuming the contrary, we have $f(x_0) > f(y^*)$. This means that an element y_n can be found such that

$$f(\sup \{y_n, x_0\}) > f(y^*) \ ;^{\star}$$

but this contradicts (1.45) since the sup $\{y_n, x_0\}$ is an element of the form (1.44).

This completes the proof of the theorem.

In the sequel, we shall call the cone K *strongly minihedral* if every bounded set M has a least upper bound sup M.

It follows from Theorem 1.15 that in separable spaces, a regular minihedral cone is strongly minihedral. In particular, the cone of non-negative functions in the spaces $L_p (1 \leqslant p < \infty)$ is strongly minihedral.

The statements, which were proved above, about the existence of a least upper bound are obviously valid in the case when the greatest lower bound (infimum) is considered.

1.7.4. *Additional remarks*

In connection with the study of minihedral cones, the question about the relation between the concepts of the minihedrality and the normality of a cone arises. As we know, the minihedral cones of non-negative functions in the spaces C and L_p are normal. It turns out that minihedral cones exist which do not possess the property of normality.

For example, let us consider in the space l_2 of numerical sequences

$$x = \{x_1, x_2, \ldots, x_n, \ldots\} \tag{1.46}$$

---

* Here, we make use of the obvious equality

$$\limsup_{n \to \infty} \{x_n, y_n\} = \sup \{\lim_{n \to \infty} x_n, \lim_{n \to \infty} y_n\},$$

which is valid under the conditions of Theorem 1.15 for arbitrary bounded non-decreasing sequences $x_n, y_n (n = 1, 2, \ldots)$.

with the norm

$$\|x\| = \{|x_1|^2 + |x_2|^2 + \ldots + |x_n|^2 + \ldots\}^{\frac{1}{2}} \tag{1.47}$$

the set K of elements whose first coordinate x_1 is non-negative and the remaining ones satisfy the inequalities

$$-\infty < x_i \leqslant x_1 \qquad (i = 2, 3, \ldots). \tag{1.48}$$

It can be verified that K is a minihedral cone which does not possess the property of normality. (We leave the appropriate calculations to the reader.)

We note further that a strongly minihedral normal cone cannot be regular. The cone of sequences

$$x = \{\xi_1, \xi_2, \ldots, \xi_n, \ldots\} \tag{1.49}$$

with non-negative components ξ_i in the space m of all bounded sequences (1.49) with the norm

$$\|x\|_m = \sup_i |\xi_i|$$

can serve as an example.

1.8. A space with two cones

1.8.1. *The boundedness of the sets* $K\langle\theta, x\rangle$

In a number of cases, it is convenient to consider two cones K and K_0, one of which is a subset of the other, in a single space E.

We have already encountered such a situation. For example, $K_{u_0} \subset K$; the cones $K_{u_0,\varrho}$ are subsets of the cone K_{u_0}; if $F_1 \subset F_2$, then $K(F_1) \subset K(F_2)$.

In the present section, it is assumed that $K_0 \subset K$. The partial ordering is introduced by means of the "broad" cone K. If it is necessary to simultaneously consider the partial ordering which is generated by the "narrow" cone K_0, then we shall denote it by the sign $\overset{\circ}{\leqslant}$.

It is not difficult to see that the normality of the cone K implies the normality of the cone K_0; the regularity and the full regularity of the cone K imply the regularity and full regularity of the cone K_0 respectively. Conversely, if the cone K_0 is solid or reproducing, then the cone K also possesses the corresponding property.

We shall denote by $K_0\langle x_0, y_0\rangle$ the collection of elements $x \in K_0$ such that $x_0 \leqslant x \leqslant y_0$. If the cone K is normal, then the set $K_0\langle x_0, y_0\rangle$ is bounded in norm for arbitrary $x_0, y_0 \in E$.

If the cone K_0 is normal and K does not possess the property of normality, then the sets $K_0\langle x_0, y_0\rangle$ might not be bounded. The cones K and K_0 in the space l_2 of sequences (1.46), the first of which is defined by the conditions $x_1 \geqslant 0$ and (1.48) and the second is defined by the conditions $x_1 \geqslant 0$ and

$$0 \leqslant x_i \leqslant x_1 \qquad (i=2, 3, \ldots), \tag{1.50}$$

can serve as an example; the cone K_0 is obviously normal, but the set $K_0\langle \theta, x_0\rangle$, where $x_0=\{2, 0, 0, \ldots\}$, is bounded in norm since the elements $\{1, 0, 0, \ldots\}$, $\{1, 1, 0, \ldots\}$, $\{1, 1, 1, \ldots\}$,... belong to $K_0\langle \theta, x_0\rangle$ and their norms (1.47) are not bounded.

Theorem 1.16. *A necessary and sufficient condition for the boundedness of the sets $K_0\langle x_0, y_0\rangle$ for arbitrary $x_0, y_0 \in K\langle x_0 \leqslant y_0\rangle$ is that the norms be semi-monotonic on K_0, that is, that $x \leqslant y (x, y \in K)$ imply that*

$$\|x\| \leqslant L\|y\| . \tag{1.51}$$

Proof. The sufficiency is obvious. We shall prove the necissity by contradiction. Suppose sequences $x_n, y_n \in K_0$ exist such that

$$x_n \leqslant y_n, \quad \|x_n\| \geqslant n^3\|y_n\| \qquad (n=1,2, \ldots).$$

The series

$$\psi = \frac{y_1}{\|y_1\|} + \frac{y_2}{4\|y_2\|} + \ldots + \frac{y_n}{n^2\|y_n\|} + \ldots$$

converges in norm. It is obvious that $\psi \in K_0$ and $\theta \leqslant x_n/n^2\|y_n\| \leqslant \psi$ $(n=1, 2, \ldots)$, where $\|x_n/n^2\|y_n\|\| \leqslant n$. This means that the set $K_0\langle \theta, \psi\rangle$ is not bounded.
This completes the proof of the theorem.

If $y-x \in K_0$, then we have, a fortiori, that $y-x \in K$. Therefore, it follows from the boundedness of all the sets $K_0\langle \theta, y_0\rangle$ that the norm on K_0 is semi-monotonic with respect to the partial ordering $\overset{\circ}{\leqslant}$. By virtue of Theorem 1.2, the cone K_0 is normal. Hence, the *normality of the cone K_0 is a necessary condition of the boundedness of all the sets $K_0\langle x_0, y_0\rangle$.*

1.8.2. *K-normality of the cone K_0*
We shall call the cone K_0 *K-normal* if

$$\|f_1^0 + f_2\| \geqslant \delta \max\{\|f_1^0\|, \|f_2\|\} \qquad (\delta > 0) \tag{1.52}$$

provided $f_1^0 \in K_0$, $f_2 \in K$, $f_1^0 + f_2 \in K_0$.

Theorem 1.17. *A necessary and sufficient condition for the K-normality of a cone K_0 is that the norm on K_0 be semi-monotonic.*

Proof. Suppose there exist sequences $f_n^0 \in K_0$ and $f_n \in K$ such that $f_n^0 + f_n \in K_0$ and

$$\|f_n^0 + f_n\| \leqslant \frac{1}{n^3} \max \{\|f_n^0\|, \|f_n\|\} = \frac{1}{n^3} a_n . \tag{1.53}$$

Since one of the numbers $\|f_n^0/a_n\|$, $\|f_n/a_n\|$ is equal to 1, we have

$$1 - \frac{1}{n^3} \leqslant \left\| \frac{f_n^0}{a_n} \right\|, \quad \left\| \frac{f_n}{a_n} \right\| \leqslant 1 + \frac{1}{n^3} \qquad (n = 1, 2, \ldots) . \tag{1.54}$$

Therefore, the inequalities

$$\tfrac{1}{2}\|f_n^0\| \leqslant \|f_n\| \leqslant 2\|f_n^0\| \qquad (n = 2, 3, \ldots)$$

are valid, from which it follows that

$$\|f_n^0 + f_n\| \leqslant \frac{2}{n^3} \|f_n^0\| \qquad (n = 2, 3, \ldots) .$$

The last inequalities mean that the series

$$f^* = \sum_{n=2}^{\infty} \left(\frac{n f_n^0}{\|f_n^0\|} + \frac{n f_n}{\|f_n^0\|} \right) \tag{1.55}$$

converges strongly. It is obvious that $f^* \in K_0$ and $\theta \leqslant n f_n^0 / \|f_n^0\| \leqslant f^*$. Therefore, the norm on K_0 does not possess the property of semi-monotonicity. This means that the K-normality of the cone follows from the semi-monotonicity of the norm.

Let the cone K_0 possess the property of K-normality. Let $x, y \in K_0$ and $x \leqslant y$. If $f_1^0 = \frac{1}{2}(y + x)$ and $f_2 = \frac{1}{2}(y - x)$, then $f_1^0 \in K_0, f_2 \in K$ and $f_1^0 + f_2 \in K_0$. Therefore

$$\|y\| = \|f_1^0 + f_2\| \geqslant \delta \max \{\|f_1^0\|, \|f_2\|\} .$$

On the other hand,

$$\|x\| = \|f_1^0 - f_2\| \leqslant \|f_1^0\| + \|f_2\| \leqslant 2 \max \{\|f_1^0\|, \|f_2\|\} .$$

Therefore,

$$\|x\| \leqslant \frac{2}{\delta} \|y\| .$$

This completes the proof of the theorem.

It is not difficult to show that every locally compact cone K_0 is K-normal, whatever the broad cone K is. The cone K_0 of all non-negative convex

functions, vanishing at the endpoints of the closed interval [0, 1] with the cone K of all non-negative functions, can serve as a non-trivial example of a K-normal cone K_0 in the space $C_1[0, 1]$ of continuously differentiable functions with norm

$$\|x(t)\|_{C_1} = \max_{0\leqslant t\leqslant 1} |x(t)| + \max_{0<t\leqslant 1} |x'(t)| .$$

1.8.3. *K-reproducing cones*

We shall call the cone K_0 *K-reproducing* if an element $u\in K_0$ can be found such that $x\leqslant u$ for arbitrary $x\in E$. In other words, the cone K_0 is called K-reproducing if every element $x\in E$ allows the representation $x=u-v$, where $u\in K_0$ and $v\in K$.

It follows directly from the definition that a reproducing cone K_0 will be K-reproducing whatever the cone K containing K_0 is. However, a K-reproducing cone might not be reproducing. The cone K_0 of non-negative non-decreasing functions in the space C of functions which are continuous on [0, 1] with the cone K of all non-negative functions can serve as an example.

1.8.4. *K-regular and fully K-regular cones*

We shall call a cone K_0 *K-regular* if every non-decreasing sequence $x_n\in K_0$, which is bounded by an element of K_0 (with respect to the partial ordering $\leqslant$), converges in norm to some element $x^*\in K_0$.

We shall call a cone K_0 *fully regular* if every non-decreasing sequence of elements of K_0 which is bounded in norm converges in norm.

We first note the obvious fact. If K is regular or fully regular, then the cone $K_0\subset K$ will be K-regular or fully K-regular, respectively. In turn, the regularity or full regularity of a cone K_0 implies the K-regularity or the full K-regularity of the cone K respectively.

Theorem 1.18. *Every K-regular and every fully K-regular cone K_0 is a K-normal (and so a fortiori a normal) cone.*

Proof. Let us assume that the cone K_0 does not possess the property of K-normality. By virtue of Theorem 1.17, sequences $x_n, u_n\in K_0$ can be found such that $x_n\leqslant u_n$ and $\|x_n\| > n^2\|u_n\|$. The series

$$v_0 = \sum_{n=1}^{\infty} \frac{u_n}{\|x_n\|}$$

converges strongly. The elements

$$y_n = \sum_{k=1}^{n} \frac{x_k}{\|x_k\|}$$

obviously satisfy the relations $y_n \in K_0$ and

$$y_1 \leqslant y_2 \leqslant \ldots \leqslant y_n \leqslant \ldots \leqslant v_0 .$$

In this connection, the sequence y_n does not converge in norm since $\|y_{n+1} - y_n\| = 1$. We have shown that K-regular cones are K-normal.
Again, let us assume that the cone K_0 does not possess the property of K-normality. Then, sequences $f_n^0 \in K_0$ and $f_n \in K$ can be found such that $f_n^0 + f_n \in K_0$ and

$$\|f_n^0 + f_n\| < \frac{1}{n^2} \max\{\|f_n^0\|, \|f_n\|\} \qquad (n = 1, 2, \ldots) .$$

It follows from the last inequalities that

$$\|f_n^0 + f_n\| < \frac{2}{n^2} \|f_n^0\| \qquad (n = 2, 3, \ldots)$$

(see the proof of Theorem 1.17). The elements

$$z_1 = \frac{f_2^0}{\|f_2^0\|}, \quad z_2 = \frac{f_2^0 + f_2}{\|f_2^0\|}, \quad z_3 = z_2 + \frac{f_3^0}{\|f_3^0\|}, \quad z_4 = z_2 + \frac{f_3^0 + f_3}{\|f_3^0\|}, \ldots$$

do not decrease and their norms are uniformly bounded:

$$\|z_n\| \leqslant 1 + 2 \sum_{1}^{\infty} \frac{1}{k^2} \qquad (n = 1, 2, \ldots) .$$

In this connection, the sequence z_n does not converge in norm since $\|z_{2k+1} - z_{2k}\| = 1$. We have shown that the K-normality of the cone follows from its full K-regularity.
This completes the proof of the theorem.

Theorem 1.19. *Every fully K-regular cone is K-regular.*

Proof. Let the sequence x_n be non-decreasing and bounded by some element of K_0. If the cone K_0 is fully K-regular, then, by Theorem 1.18, it is K-normal and, by Theorem 1.17, the sequence x_n is bounded in norm. Consequently, the sequence x_n converges in norm.
This completes the proof of the theorem.

Not every K-regular cone K_0 is fully K-regular. For the case $K = K_0$, examples were given in the preceding sections. Examples can also be given for the case when $K \neq K_0$.
V. Ya. Stecenko gave several criteria, which, when satisfied, yield that

full K-regularity follows from K-regularity. For this it is sufficient that the cone K_0 be solid (the proof is analogous to the proof of Theorem 1.8); it is sufficient that the space E be regular; it is sufficient that the cone K be normal and the space E weakly complete (the proof is analogous to the proof of Theorem 1.9). We shall not consider other criteria.

1.8.5. *Weakly regular cones*

Further generalizations of the concepts of a regular cone are associated with weak convergence.

We consider the space E with one cone K. We shall call the cone K *weakly regular* if

$$x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n \leqslant \ldots \qquad (x_n \in K) \tag{1.56}$$

and

$$x_n \leqslant u \qquad (n = 1, 2, \ldots) \tag{1.57}$$

imply the weak convergence of the sequence x_n. We shall call the cone K *weakly fully regular* if it is normal and if (1.56) and

$$\|x_n\| \leqslant M \qquad (n = 1, 2, \ldots) \tag{1.58}$$

imply the weak convergence of the sequence x_n.

The theory of weakly regular cones was developed by V. Ya. Stecenko [4]. In particular, he showed that every weakly fully regular cone is weakly regular. (The proof is analogous to the proof of Theorem 1.7.) A regular and weakly fully regular cone is fully regular. It turns out that every weakly regular cone is normal.

The concepts connected with the weak regularity of a cone generalize in a natural way to the case of spaces with two cones.

Chapter 2

LINEAR POSITIVE OPERATORS

In this chapter, we shall study linear operators which leave some cone invariant in a Banach space. Classes of operators are chosen having characteristic vectors in the cone. Under general assumptions, such characteristic vectors turn out to be unique (to within the norm) and the corresponding characteristic values are greater than the absolute magnitudes of the remaining characteristic values.

2.1. Linear u_0-positive operators

2.1.1. *Definitions*

Let E be a Banach space with the cone K. The linear operator A is called *positive* if it transforms the cone K into itself: it follows from $x \geqslant \theta$ that $Ax \geqslant \theta$.

It is not difficult to see that the positive linear operator A possesses the property of *monotoneity*: it follows from $x \leqslant y$ for arbitrary elements $x, y \in E$ that $Ax \leqslant Ay$. In fact, $x \leqslant y$ means that $y - x \geqslant \theta$; but then $Ay - Ax \geqslant \theta$.

We denote by u_0 some fixed non-zero element of K. We call the linear positive operator A *u_0-bounded below* if for every non-zero $x \in K$ a natural number n and a positive number α can be found such that

$$\alpha u_0 \leqslant A^n x .$$

Analogously, we call the operator A *u_0-bounded above* if for every non-zero $x \in K$ an m and a β can be found such that

$$A^m x \leqslant \beta u_0 .$$

Finally, if for every $x \in K (x \neq \theta)$

$$\alpha u_0 \leqslant A^n x \leqslant \beta u_0$$

for some n, then we call the operator A *u_0-positive*.

If the cone K is solid and for every non-zero x of K an n can be found such

that $A^n x$ is an interior element of the cone, then the operator A is called *strongly positive*. Strongly positive operators are the simplest examples of u_0-positive operators where u_0 is an arbitrary interior element of the cone K. In fact, if $A^n x$ is an interior element of the cone, then for a sufficiently small $\alpha > 0$ the element $A^n x - \alpha u_0$ also will be an element of the cone, i.e. $\alpha u_0 \leqslant A^n x$. Analogously, if u_0 is an interior element of the cone, then for sufficiently large β the element $u_0 - (1/\beta)A^n x$ also belongs to the cone, i.e. $A^n x \leqslant \beta u_0$.

The above arguments also show that every positive operator is u_0-bounded above if the cone K is solid and u_0 is an interior element of this cone.

2.1.2. *Examples*

In the case of finite-dimensional spaces with a cone consisting of vectors with non-negative components, the linear positive operators are defined by matrices with non-negative elements.

Of fundamental interest to us will be linear integral operators

$$Ax(t) = \int_\Omega k(t, s)x(s)\,ds\,, \tag{2.1}$$

acting in various spaces E of functions defined on the set Ω which, everywhere in the sequel, we assume to be a bounded and closed subset of a finite-dimensional space. It is assumed that the kernel satisfies the conditions for which the operator (2.1) acts and is continuous in a corresponding space.

If the set of all non-negative functions from corresponding spaces (C, L_p, etc.) is considered as a cone, then the operator (2.1) will evidently be positive if the kernel $k(t, s)$ is non-negative:

$$k(t, s) \geqslant 0 \qquad (t, s \in \Omega)\,.$$

If the operator (2.1) acts in the space C and if some iteration

$$k_N(t, s) = \int_\Omega \dots \int_\Omega k(t, \sigma_1)k(\sigma_1, \sigma_2) \dots k(\sigma_{N-1}, s)\,d\sigma_1\,d\sigma_2 \dots d\sigma_{N-1} \tag{2.2}$$

is positive, then the operator (2.1) will be strongly positive in C since the function $A^N x(t)$ will be a positive function whatever the non-negative, but not identically equal to zero, function $x(t)$ is.

We assume that the operator (2.1) acts in L_p. If, in this connection, the kernel $k(t, s)$ satisfies the inequality

$$k(t, s) \geqslant m > 0 \qquad (t, s \in \Omega), \tag{2.3}$$

then the operator (2.1) will be u_0-bounded below if the function $u_0(t) \equiv 1$ is chosen as u_0. This follows from the obvious inequality

$$Ax(t) \geqslant m \int_\Omega x(s)\,ds = m \int_\Omega x(s)\,ds\, u_0(t).$$

The operator (2.1), with condition (2.3), does not necessarily have the property of u_0-boundedness above. For example, if $k(t, s) = k_0(t)$ where $k_0(t) \geqslant m$ and $k_0(t)$ is not bounded, then the functions $A^n x(t)$ will not be u_0-bounded above and, consequently, A will not be u_0-positive if $u_0(t) \equiv 1$. In the study of integral operators (2.1) with non-negative kernels, several invariant cones can be shown which are different from the cone of all non-negative functions. For example, let the kernel $k(t, s)$ be non-negative and satisfy the condition

$$u_0(t) \leqslant k(t, s) \leqslant \varrho u_0(t) \qquad (t, s \in \Omega), \tag{2.4}$$

where $\varrho \geqslant 1$, and the function $u_0(t)$ belongs to the space (C, L_p, etc.) in which the operator (2.1) acts. It is obvious that

$$\int_\Omega x(s)\,ds\, u_0(t) \leqslant Ax(t) \leqslant \varrho \int_\Omega x(s)\,ds\, u_0(t)$$

for every non-negative function $x(t)$.

Consequently, the operator (2.1) with the condition (2.4) leaves the cone $K_{u_0, \varrho}$ invariant where K is the cone of all non-negative functions. Moreover, we have shown that the operator (2.1), with the condition (2.4), transforms the entire cone K into a proper subset of it in the cone $K_{u_0, \varrho}$.

Theorem 2.1. *Let the linear operator A transform the cone K into the cone $K_{u_0, \rho}$ where u_0 is some non-zero element of K and let it follow from $Ax = \theta$ ($x \in K$) that $x = \theta$. Then the operator A will be u_0-positive on every cone $K_{u_0, \rho+\varepsilon}$.*

Proof. The partial ordering defined by the cone K will be denoted by the sign $\leqslant$ and the sign $\overset{(\varepsilon)}{\leqslant}$ will denote the partial ordering generated by the cone $K_{u_0, \varrho+\varepsilon}$ where ε is a fixed non-negative number.

Let $x \in K_{u_0, \varrho+\varepsilon}$, $x \neq \theta$. Then $Ax \in K_{u_0, \varrho}$; that is, an $a > 0$ can be found such that

$$au_0 \leqslant Ax \leqslant \varrho a u_0. \tag{2.5}$$

We set

$$a_0 = \frac{\varrho - 1}{\varrho - 1 + \varepsilon}\, a, \quad a = a - a_0 = \frac{\varepsilon a}{\varrho - 1 + \varepsilon}, \quad \beta = \rho a + a_0.$$

Then the inequalities

$$a_0u_0 \leqslant Ax - \alpha u_0 \leqslant (\varrho + \varepsilon)a_0u_0$$

and

$$a_0u_0 \leqslant \beta u_0 - Ax \leqslant (\varrho + \varepsilon)a_0u_0$$

follow from (2.5).
The inequalities thus obtained mean that

$$\alpha u_0 \overset{(\varepsilon)}{\leqslant} Ax \overset{(\varepsilon)}{\leqslant} \beta u_0 .$$

This completes the proof of the theorem.

We shall stress that Theorem 2.1 applies not only to operators of the form (2.1).

2.1.3. *Positiveness of an operator which is bounded above and below*
It does not follow directly from the definitions that an operator A which is u_0-bounded above and below is a u_0-positive operator since for every non-zero $x \in K$ the inequalities $\alpha u_0 \leqslant A^n x$ and $A^m x \leqslant \beta u_0$ can be satisfied for different numbers n and m. However, the following theorem holds.

Theorem 2.2. *If the operator A is u_0-bounded above and below, then it is u_0-positive.*

Proof. In the first place, we note that for some natural number k the inequalities

$$\alpha_0u_0 \leqslant A^k u_0 \leqslant \beta_0 u_0 \tag{2.6}$$

are satisfied. As k it suffices, for example, to choose the product $k=st$, where $\alpha_1 u_0 \leqslant A^s u_0$ and $A^t u_0 \leqslant \beta_1 u_0$. Then the inequalities (2.6) will be satisfied for $\alpha_0 = \alpha_1{}^t$ and $\beta_0 = \beta_1{}^s$.
Let x be a fixed non-zero element of K. We denote by m and n natural numbers such that

$$\alpha u_0 \leqslant A^n x , \quad A^m x \leqslant \beta u_0 \qquad (\alpha, \beta > 0) . \tag{2.7}$$

Without loss of generality, we can assume that $n > m$. It then follows from (2.7) that

$$A^{n-m}u_0 \geqslant A^{n-m}\left(\frac{1}{\beta}A^m x\right) = \frac{1}{\beta}A^n x \geqslant \frac{\alpha}{\beta}u_0 . \tag{2.8}$$

We first assume that the numbers $n-m$ and k are relatively prime. Then positive natural numbers p and q can be found such that either

$$p(n-m)-qk=1\,, \tag{2.9}$$

or

$$-p(n-m)+qk=1\,. \tag{2.10}$$

In the first case, by virtue of (2.6) and (2.8),

$$Au_0=A^{p(n-m)-qk}u_0\geqslant A^{p(n-m)-qk}\left(\frac{1}{\beta_0{}^q}A^{qk}u_0\right)=$$

$$=\frac{1}{\beta_0{}^q}A^{p(n-m)}u_0\geqslant\frac{1}{\beta_0{}^q}\left(\frac{\alpha}{\beta}\right)^p u_0\,. \tag{2.11}$$

Let $m+rk>n$. Then, by virtue of (2.6) and (2.7),

$$A^{m+rk}x\leqslant\beta A^{rk}u_0\leqslant\beta\beta_0{}^r u_0$$

and, by virtue of (2.11),

$$A^{m+rk}x\geqslant\alpha A^{m+rk-n}u_0\geqslant\alpha\left[\frac{1}{\beta_0{}^q}\left(\frac{\alpha}{\beta}\right)^p\right]^{m+rk-n}u_0\,.$$

Thus, in the case of (2.9), the element $A^{m+rk}x$ will satisfy the inequalities $\alpha' u_0\leqslant A^{m+rk}x\leqslant\beta' u_0$ ($\alpha', \beta'>0$).

We consider the case (2.10). Again by virtue of (2.6) and (2.8),

$$Au_0=A^{qk-p(n-m)}u_0\leqslant A^{qk-p(n-m)}\left[\left(\frac{\beta}{\alpha}\right)^p A^{p(n-m)}u_0\right]=$$

$$=\left(\frac{\beta}{\alpha}\right)^p A^{qk}u_0\leqslant\left(\frac{\beta}{\alpha}\right)^p\beta_0{}^q u_0\,.$$

Then

$$A^n x\leqslant\beta A^{n-m}u_0\leqslant\beta\left[\left(\frac{\beta}{\alpha}\right)^p\beta_0{}^q\right]^{n-m}u_0$$

and it follows from (2.7) that

$$\alpha u_0\leqslant A^n x\leqslant\beta' u_0\,.$$

We now assume that the numbers $n-m$ and k do not have the property of being relatively prime. Let

$$k=k_0 d\,,\quad n=l+n_0 d\,,\quad m=l+m_0 d\,,$$

where d is the greatest common divisor of the numbers k and $n-m$. It is obvious that $m_0<n_0$. We introduce the notation $A^d=B$, $A^l x=y$. Then the inequalities (2.6) and (2.7) can be rewritten in the form

$$\alpha_0 u_0\leqslant B^{k_0}u_0\leqslant\beta_0 u_0$$

and

$$\alpha u_0\leqslant B^{n_0}y\,,\quad B^{m_0}y\leqslant\beta u_0\,.$$

Since now the numbers n_0-m_0 and k_0 are relatively prime, then, according

to what has already been proved, an s can be found such that

$$\alpha' u_0 \leqslant B^s y \leqslant \beta' u_0 \qquad (\alpha', \beta' > 0) .$$

The last inequality can be rewritten in the form

$$\alpha' u_0 \leqslant A^{sd+l} x \leqslant \beta' u_0 .$$

This completes the proof of the theorem.

2.1.4. *Continuity of a positive operator*

The arguments considered above were of a purely algebraic character and were not connected with the continuity of the operator. We give some general propositions concerning the continuity of positive operators.

Theorem 2.3. *Let E_1 and E_2 be two Banach spaces with cones K_1 and K_2 respectively, the first of which is solid and the second normal. Let an additive homogeneous operator A which is positive in the sense that $AK_1 \subset K_2$ act from E_1 to E_2.*
Then the operator A is continuous.

Proof. Let u_0 be an interior element of the cone K_1. Then the set $\langle \theta, u_0 \rangle$ of all the elements $x \in K_1$ which satisfy the inequalities $\theta \leqslant x \leqslant u_0$ will contain some sphere. Therefore, it suffices to prove the boundedness of the set $A\langle \theta, u_0 \rangle$ in the norm of the space E_2.
If $\theta \leqslant x \leqslant u_0$, then $\theta \leqslant Ax \leqslant Au_0$. By virtue of Theorem 1.2, the norm in E_2 is semi-monotonic and it follows from the last inequalities that

$$\|Ax\|_{E_2} \leqslant N \|Au_0\|_{E_2} \qquad (x \in \langle \theta, u_0 \rangle) .$$

This completes the proof of the theorem.*

The proof remains valid if the spaces E_1 and E_2 are not complete. In this case, the operator A can be extended by continuity to complete spaces. Since, by virtue of Theorem 1.4, the cone $\bar{K}_{u_0}$ is solid and normal, an additive and homogeneous operator, transforming K_{u_0} into itself, is continuous in the u_0-norm.

2.1.5. *Uniformly positive operators*

We call the linear operator which leaves the cone K invariant *uniformly*

* The theorem remains valid if the assumption about the solidity of the cone K_1 is replaced by the less restrictive assumption that the cone K_1 is reproducing. (See I. A. Bakhtin, M. A. Krasnosel'skii, V. Ya. Stecenko [1].)

positive if

$$\|Ax\| \geqslant b\|x\| \qquad (x \in K).$$

The unit vector I:

$$Ix \equiv x \qquad (x \in E)$$

can serve as an example of a uniformly positive operator.

Theorem 2.4. *If a uniformly positive linear completely continuous operator A is defined on the cone K, then K allows plastering.*

Proof. We introduce the notation $K_0 = \overline{AK}$. It is obvious that K_0 is a cone. This cone K_0 is locally compact (see p. 33) since its intersection with the unit sphere lies in the closure of the compact set of the values of the operator A on the intersection of the cone K with the sphere of radius $1/b$. As shown above, it follows from the local compactness of the cone K_0 that K_0 allows plastering.

We denote by $f_0(x)$ a linear uniformly positive functional on K_0:

$$f_0(x) \geqslant a_0\|x\| \qquad (x \in K_0).$$

We set $f(x) = f_0(Ax)$. The linear functional $f(x)$ is uniformly positive on K:

$$f(x) = f_0(Ax) \geqslant a_0\|Ax\| \geqslant a_0 b\|x\| \qquad (x \in K).$$

It follows from Theorem 1.5 that K allows plastering.

This completes the proof of the theorem.

It is natural to supplement this theorem with the remark that on every cone K allowing plastering a uniformly positive linear completely continuous operator A can be defined. For example, it suffices to set $Ax = f(x)u_0$, where u_0 is some non-zero element of K and $f(x)$ is a uniformly positive linear functional.

The operator

$$Ax(t) = \int_\Omega k(t, s)x(s)\,ds \tag{2.1}$$

can serve as an example of a uniformly positive linear operator, the kernel of which satisfies the condition (2.4):

$$u_0(t) \leqslant k(t, s) \leqslant \varrho u_0(t) \qquad (t, s \in \Omega).$$

If this operator acts, for example, in the space C of functions continuous on Ω and if $u_0(t) \in C$, then the operator (2.1) transforms the cone $K_{u_0,\varrho}$ of continuous functions $x(t)$ satisfying, for some $a = a[x(t)]$, the condition

$$au_0(t) \leqslant x(t) \leqslant \varrho a u_0(t) \qquad (t \in \Omega)$$

into itself, and on this cone the inequality

$$Ax(t) = \int_\Omega k(t, s)\, x(s)\, ds \geqslant u_0(t) \int_\Omega x(s)\, ds \geqslant$$

$$\geqslant a u_0(t) \int_\Omega u_0(s)\, ds \geqslant \frac{1}{\varrho} \int_\Omega u_0(s)\, ds\, x(t) \qquad (t \in \Omega)$$

is satisfied.

The last inequality means that the operator A is uniformly positive. We recall that (by virtue of Theorem 2.1) the cone $K_{u_0, \varrho}$ allows plastering. We assume that the linear (not necessarily completely continuous) operator A satisfies the condition which is close to uniform positiveness

$$Ax \geqslant m \|x\| u_0\,, \tag{2.12}$$

where u_0 is some fixed non-zero element of K. We define the positive functional $f_0(x)$ so that $f_0(u_0) = 1$, and we set $f(x) = f_0(Ax)$. It will then follow from (2.12) that $f(x)$ is a uniformly positive functional.

Thus, *it follows from the existence of the linear operator* A, *satisfying the condition* (2.12), *that the cone* K *allows plastering.*

A cone which allows plastering is normal; therefore (2.12) is a more severe restriction than is the uniform positiveness of an operator.

2.2. Existence of a characteristic vector

2.2.1. *Positive characteristic vectors*

The non-zero element $x \in E$ is called a *characteristic vector* of the linear operator A if

$$Ax = \lambda x\,,$$

where λ is some real number which we call a *characteristic value* of the operator A.

Let us assume that the linear operator A acts in the space E with the cone K and leaves this cone invariant. In other words, A is a positive operator. We shall be interested in the existence of characteristic vectors, belonging to K, of the operator A. These characteristic vectors we shall call *positive*. We shall call the characteristic values λ corresponding to positive characteristic vectors of the positive linear operator A *positive* characteristic values. It is obvious that the positive characteristic values are non-

negative. We emphasize that we do not call positive the characteristic values of the operator A to which correspond vectors not lying in K, even if they are positive.

2.2.2. *Schauder's fixed-point principle*

The (linear or non-linear) operator $\boldsymbol{B}$, acting in the space E, is called *completely continuous* if it is continuous and if it transforms every bounded set into a compact set. The concept of complete continuity generalizes in a natural way to operators which are defined on some set $T \subset E$.

In the proof of the existence of solutions of the equation

$$x = \boldsymbol{B}x$$

with a completely continuous operator $\boldsymbol{B}$, it is convenient in many cases to make use of the well-known Schauder principle:* *if a completely continuous operator $\boldsymbol{B}$ transforms a bounded convex and closed set T into itself, then the operator $\boldsymbol{B}$ has at least one fixed point x^* on T.* The fixed point x^* of the operator $\boldsymbol{B}$ is a solution of the equation $x = \boldsymbol{B}x$.

2.2.3. *The existence of a positive characteristic vector for a linear completely continuous operator*

Theorem 2.5. *Let the linear positive operator $\boldsymbol{A}$ be completely continuous. Let the relation*

$$\boldsymbol{A}^p u \geqslant \alpha u \qquad (\alpha > 0) \tag{2.13}$$

be satisfied for some non-zero element u such that $-u \in K$, $u = v - w$ $(v, w \in K)$ where p is some natural number.

Then the operator $\boldsymbol{A}$ has at least one characteristic vector x_0 in K:

$$\boldsymbol{A}x_0 = \lambda_0 x_0 , \tag{2.14}$$

where the positive characteristic number λ_0 satisfies the inequality

$$\lambda_0 \geqslant \sqrt[p]{\alpha} . \tag{2.15}$$

Proof. We shall prove the existence of a single characteristic vector $x_0 (\|x_0\| = 1)$ for the operator $\boldsymbol{A}$ in the cone K.

We denote by $\boldsymbol{A}_n (n = 1, 2, \ldots)$ the operators defined by the equality

* See L. V. Kantorovič and G. P. Akilov [1].

$$A_n x = \frac{A\left(x + \frac{v}{n}\right)}{\left\| A\left(x + \frac{v}{n}\right) \right\|}, \tag{2.16}$$

and we consider them on the intersection T of the cone K with the unit sphere $\|x\| \leqslant 1$. Every one of the operators A_n transforms T into itself and is completely continuous together with the operator A (it suffices to observe that

$$A\left(x + \frac{v}{n}\right) \geqslant \frac{1}{n} Av \neq \theta ,$$

by virtue of which

$$\left\| A\left(x + \frac{v}{n}\right) \right\| > 0 \quad \text{for} \quad x \in T \Big) .$$

By virtue of the Schauder principle, every operator A_n has a fixed point $x_n \in T$. It follows from (2.16) that $\|x_n\| = 1$. We rewrite the equalities in the form

$$A\left(x_n + \frac{v}{n}\right) = \lambda_n x_n \qquad (n = 1, 2, \ldots) . \tag{2.17}$$

The relations

$$x_n \geqslant \frac{1}{\lambda_n} Ax_n , \quad x_n \geqslant \frac{1}{n\lambda_n} Av \qquad (n = 1, 2, \ldots) \tag{2.18}$$

follow from (2.17). It follows from this and from condition (2.13) that

$$x_n \geqslant \frac{1}{\lambda_n^{p-1}} A^{p-1} x_n = \frac{1}{\lambda_n^p} A^p \left(x_n + \frac{v}{n}\right) \geqslant \frac{1}{n\lambda_n^p} A^p v \geqslant$$
$$\geqslant \frac{1}{n\lambda_n^p} A^p u \geqslant \frac{\alpha}{n\lambda_n^p} u \qquad (n = 1, 2, \ldots) .$$

Therefore, there exists a maximal $\beta_n > 0$ such that $x_n \geqslant \beta_n u$. Further, the relations

$$x_n \geqslant \frac{1}{\lambda_n^p} A^p \left(x_n + \frac{v}{n}\right) \geqslant \frac{1}{\lambda_n^p} \left(\beta_n + \frac{1}{n}\right) A^p u \geqslant \frac{\alpha}{\lambda_n^p} \left(\beta_n + \frac{1}{n}\right) u \tag{2.19}$$

follow from (2.18) and (2.13).
The inequalities

$$\frac{\alpha}{\lambda_n^p} \left(\beta_n + \frac{1}{n}\right) \leqslant \beta_n \qquad (n = 1, 2, \ldots)$$

follow from the maximality of the numbers β_n and these inequalities can

be rewritten in the form

$$\lambda_n^p \geqslant \alpha + \frac{\alpha}{n\beta_n} \qquad (n=1, 2, \ldots). \tag{2.20}$$

Since the operator A is completely continuous, a subsequence of indices $n_i (i=1, 2, \ldots)$ can be chosen such that the sequence $A[x_{n_i}+(v/n_i)]$ strongly converges to some element y^*. By virtue of (2.20), with this choice of the sequence n_i the convergence of the numbers λ_{n_i} to some number λ_0 which satisfies the inequality (2.15) can be guaranteed simultaneously. Then the elements x_{n_i} will converge in norm to the element $x_0=y^*/\lambda_0$. To obtain the equality (2.14), it suffices to pass to the limit in the equalities

$$A\left(x_{n_i} + \frac{v}{n_i}\right) = \lambda_{n_i} x_{n_i} \qquad (i=1, 2, \ldots).$$

This completes the proof of the theorem.

The theorem proved contains two essential restrictions: condition (2.13) and the assumption of complete continuity of the operator A. In the following subsections, cases will be considered when these restrictions relative to the operator A can be replaced by assumptions about some properties of the space E and the cone K.

2.2.4. *Existence of a characteristic vector for a uniformly positive operator*

We recall that the linear positive operator A is called uniformly positive if

$$\|Ax\| \geqslant b\|x\| \qquad (x \in K),$$

where $b>0$.

Theorem 2.6. *Let a completely continuous linear operator A be uniformly positive.*

Then A has at least one positive characteristic vector.

Proof. We introduce for consideration the operator

$$Bx = \frac{Ax+(1-\|x\|)u_0}{\|Ax+(1-\|x\|)u_0\|} \qquad (x \in K,\ \|x\| \leqslant 1),$$

where u_0 is some fixed non-zero element of K.

The operator B is completely continuous on the intersection T of the cone K with the unit sphere $\|x\| \leqslant 1$. Its continuity follows from the fact that the elements

$$Ax+(1-\|x\|)u_0 \qquad (x \in T)$$

are different from the zero θ. The compactness of the set BT follows from

the complete continuity and uniform positiveness of the operator A. In fact, a subsequence $x_n (n=1, 2, \ldots)$ can be selected from an arbitrary sequence of elements of the set T in order that the elements Ax_n converge to some element y^* and the numbers $\|x_n\|$ converge to some non-negative number α. The numbers

$$\beta_n = \|Ax_n + (1 - \|x_n\|)u_0\|$$

will, in this connection, converge to some limit β where $\beta > 0$. The last inequality follows from the uniform positiveness of the operator A if $\alpha = 1$ since

$$\|Ax_n + (1 - \|x_n\|)u_0\| \geqslant \|Ax_n\| - (1 - \|x_n\|)\|u_0\| \geqslant$$
$$\geqslant b\|x_n\| - (1 - \|x_n\|)\|u_0\| .$$

If $\alpha < 1$ then the elements $Ax_n + (1 - \|x_n\|)u_0$ converge to a non-zero element $y^* + (1-\alpha)u_0$, by virtue of which $\beta = \|y^* + (1-\alpha)u_0\| > 0$. Thus, a subsequence x_n can be chosen from an arbitrary sequence of elements of the set T in order that the elements Bx_n strongly converge to some limit. Consequently, BT is compact.

The completely continuous operator B transforms the set T into itself. By virtue of the Schauder principle, the operator B has a fixed point x^* where

$$\|x^*\| = \|Bx^*\| = 1 .$$

But then

$$Ax^* = \|Ax^*\|x^* .$$

This completes the proof of the theorem.

In the conditions of Theorem 2.6, restrictions in an explicit form are not placed on the cone K which the operator A leaves invariant. However, the uniform boundedness of the completely continuous operator A means, by virtue of Theorem 2.5, that the cone K allows plastering. Since the cones of the non-negative functions in the space C of continuous functions and in the spaces $L_p (1 < p < \infty)$ do not allow plastering, then in applications, as a rule, Theorem 2.4 is more convenient in spite of condition (2.13).

2.2.5. *Weak topology and the Tikhonov-Schauder principle*

We call the operator A *weakly continuous* if it transforms every weakly convergent sequence of elements into a sequence which also converges weakly.

As is well known, every linear continuous operator A is weakly contin-

uous. In fact, let the sequence x_n weakly converge in itself. Then for an arbitrary linear functional $f(x)$ the sequence $f(Ax_n)$ converges since $f(Ax_n)=A^*f(x_n)$ $(n=1, 2, \ldots)$ where A^* is the linear operator adjoint to A.

We recall that the set $T\subset E$ is called *weakly compact* if a weakly convergent subsequence can be selected from every sequence $x_n\in T$. The unit sphere in the spaces $L_p(1<p<\infty)$ is weakly compact. If the unit sphere in the space is weakly compact, then every bounded closed convex set will be weakly compact.

The Tikhonov [1] principle about the existence of a fixed point will be utilized below: *if a weakly continuous operator* $\boldsymbol{B}$ *transforms a weakly compact and weakly complete set* T *into itself, then the operator* $\boldsymbol{B}$ *has at least one fixed point on* T.

2.2.6. *Linear operators in weakly complete spaces*

Theorem 2.7. *Let the space* E *be weakly complete and the unit sphere in* E *be weakly compact. Let the cone* K *allow plastering.*

Then every linear operator A, *leaving the cone* K *invariant, has at least one characteristic vector.*

Proof. Since K allows plastering, then, by virtue of Theorem 1.5, a uniformly positive linear functional exists on K, that is a functional $f_0(x)$ such that

$$f_0(x)\geqslant a\|x\| \qquad (x\in K). \tag{2.21}$$

We denote by T the set of elements x, belonging to K, such that $f_0(x)=1$. The set T is the intersection of the cone K and the hyperplane $f_0(x)=1$. Therefore T is convex and closed. It follows from (2.21) that

$$\|x\| \leqslant \frac{1}{a} \qquad (x\in T).$$

Thus, T is compact in the weak topology.

Let

$$Bx = \frac{x+Ax}{f_0(x+Ax)} \qquad (x\in T). \tag{2.22}$$

The operator $\boldsymbol{B}$ transforms T into itself.

The weak continuity of the operator (2.22) evidently follows from the weak continuity of the operator A. Consequently, a continuous operator $\boldsymbol{B}$ in the weak topology transforms the compact convex set T in the weak

topology into itself. It follows from the Tikhonov principle that the operator B has at least one fixed point $x_0 \in T$. This means that

$$Ax_0 = [f_0(x_0 + Ax_0) - 1]x_0 = f_0(Ax_0)x_0 .$$

This completes the proof of the theorem.

Any additional requirements do not arise in this theorem for the positive operator A. However, highly restricting requirements are attached to the space and the cone because, as a rule, neither the cones of the non-negative functions in usual functional spaces allow plastering nor does the unit sphere in these spaces possess the property of weak compactness. We observe that all the conditions of Theorem 2.7 are satisfied if the space E is finite-dimensional.
The example considered in the following subsection shows that both conditions of Theorem 2.7 are essential.

Theorem 2.8. *Let the space E be weakly complete and the unit sphere in the space E be weakly compact. Let the cone K be normal.*
Let the linear operator A satisfy the condition

$$a(x)u_0 \leqslant Ax \leqslant \varrho a(x)u_0 \qquad (x \in K), \tag{2.23}$$

where $a(x) \geqslant 0$, $\varrho \geqslant 1$.
Then A has at least one positive characteristic vector.

Proof. It follows from (2.23) that the operator A leaves the cone $K_{u_0,\varrho}$ invariant which, by virtue of Theorem 1.13, allows plastering. Therefore, the existence of a positive characteristic vector follows from Theorem 2.7. This completes the proof of the theorem.

2.2.7. *Examples*
We consider the operator

$$Ax(t) = \int_0^t x(s)\,ds \tag{2.24}$$

in various spaces E of functions on [0, 1]. The operator (2.24) transforms the cone of non-negative functions into itself but does not have characteristic vectors for the operator A either in a space L in which the cone allows plastering but the sphere does not possess the property of weak compactness, or in the spaces $L_p (1 < p < \infty)$ in which the sphere is weakly compact but the cone of positive functions does not allow plastering. Thus, both conditions of Theorem 2.7 are essential.

The operator (2.24) is completely continuous in the spaces $L_p(1 \leqslant p < \infty)$ as well as in the space C of continuous functions. The absence of positive characteristic functions means that it is impossible to relinquish condition (2.13) in Theorem 2.5.

We give examples of integral operators

$$Ax(t) = \int_\Omega k(t, s)x(s)ds\,, \tag{2.1}$$

satisfying the conditions of the theorems proved in this section.

If the kernel $k(t, s)$ satisfies the condition (2.4) and the operator (2.1) acts and is continuous (but not necessarily completely continuous!) in one of the spaces $L_p(1 < p < \infty)$, then the conditions of Theorem 2.8 are satisfied —the operator A has at least one non-negative characteristic function.

Let the kernel $k(t, s)$ satisfy the condition

$$k_N(t, s) \geqslant u(t) \qquad (t, s \in \Omega)\,, \tag{2.25}$$

where $k_N(t, s)$ is an iterated kernel (see (2.2)) and $u(t)$ is a non-negative function of the space in which the operator (2.1) is completely continuous. Then the conditions of Theorem 2.5 are satisfied since the inequality

$$A^N x(t) = \int_\Omega k_N(t, s)x(s)ds \geqslant \int_\Omega x(s)ds\,u(t) \qquad (t \in \Omega)$$

follows from (2.25), which can be considered as condition (2.13). Thus, the existence of at least one non-negative function for the operator (2.1) follows from (2.25).

2.2.8. *One further criterion for the existence of a characteristic vector*

We recall the definition of complex characteristic values of the linear operators A considered in the real space E.

We denote by $\tilde{E}$ the collection of ordered pairs $\{x, y\}$ where $x, y \in E$. We shall denote the pair $\{x, y\}$ by $z = x + iy$. The sum and difference of two pairs is defined by the equality

$$(x_1 + iy_1) \pm (x_2 + iy_2) = (x_1 \pm x_2) + i(y_1 \pm y_2)\,.$$

If $\lambda = \sigma + i\tau$, then the product λz is defined by the equality

$$(\sigma + i\tau)(x + iy) = (\sigma x - \tau y) + i(\tau x + \sigma y)\,.$$

It is not difficult to see that with this introduction of algebraic operations $\tilde{E}$ becomes a linear system over the field of complex numbers.

The space $\tilde{E}$ becomes a normed space if we set

$$\|x+iy\| = \max_{0\leqslant\theta\leqslant 2\pi} \|x\sin\theta + y\cos\theta\| .$$

The space $\tilde{E}$ is complete if E is complete.
The operator A is extended from a real space E to a complex extension $\tilde{E}$ by means of the equality

$$A(x+iy)=Ax+iAy \qquad (x, y\in E) .$$

The number λ (which can be complex) is called a *characteristic value* of the operator A if a non-zero element $z_0\in\tilde{E}$ can be found such that $Az_0=\lambda z_0$. In the sequel, when speaking of the characteristic values of an operator, we shall assume that we are dealing with real as well as with complex characteristic values.
Let us assume that the completely continuous operator A has a characteristic vector u, corresponding to the characteristic value μ, which is a pth root of the positive number α_0. Then

$$A^p u=\alpha_0 u ,$$

and the element u can be assumed real; that is, it is assumed that $u\in E$. Without loss of generality, we can assume that $-u\in K$. If the element u is representable in the form

$$u=v-w \qquad (v, w\in K) ,$$

then it will follow from Theorem 2.5 that the operator A has at least one characteristic vector belonging to the cone K.
It turns out that this fact allows an essential generalization.

Theorem 2.9. *Let the linear completely continuous operator A leave the cone K invariant. Let the operator A have a non-zero real or complex characteristic value to which corresponds a characteristic vector in the closure of the linear hull of the cone K.*
Then the operator A has at least one positive characteristic vector.

We shall not occupy ourselves with a full proof of this theorem (see M. G. Krein and M. A. Rutman [1]). We shall complete the proof only for the case when K is a reproducing cone.
It remains for us to consider the case when A has a complex characteristic value μ not being a pth root (p an integer) of a positive number.
We consider the operators $A+\varepsilon_n A^2$; these operators have the characteristic value $\mu+\varepsilon_n\mu^2$. We choose the sequence ε_n such that it tends to zero and that the numbers $\mu+\varepsilon_n\mu^2$ are pth roots (p an integer) of some positive

numbers. By what has already been proved, every one of the operators $A+\varepsilon_n A^2$ has a normed positive characteristic vector $h_n \in K$:

$$(A+\varepsilon_n A^2)h_n=\mu_n h_n ,$$

where it follows from (2.15) that

$$\mu_n \geqslant |\mu+\varepsilon_n \mu^2| .$$

Without loss of generality, it can be assumed that the numbers μ_n converge to some number $\mu_0 \geqslant |\mu|$ and the elements Ah_n converge to some element g_0. Passing to the limit in the equalities

$$h_n = \frac{1}{\mu_n}(Ah_n+\varepsilon_n A^2 h_n) \qquad (n=1, 2, \ldots) ,$$

we are convinced of the convergence of the sequence h_n to some element $h_0 \in K$ and of the validity of the equality $Ah_0=\mu_0 h_0$.
With this the proof is completed.

It follows from this proof that one of the positive characteristic values of the completely continuous positive linear operator A is not smaller than the moduli of the remaining characteristic values.

2.2.9. *The existence of a characteristic vector in a narrow cone*
Cases were considered above when the operator A leaves two cones K_0 and K invariant simultaneously where $K_0 \subset K$. As a rule, in such cases, it is desirable to prove the existence of a characteristic vector in a "narrow" cone K_0.
If the cone K allows plastering, then the cone K_0 also allows plastering. Therefore, under the conditions of Theorem 2.7, it can be stated that A has a characteristic vector in K_0 if $AK_0 \subset K_0$.
It turns out that under the conditions of Theorem 2.6, the operator A has at least one characteristic vector in the "narrow" cone K_0 if $AK_0 \subset K_0$ and $v \in K_0$. In order that we can be convinced of this, it suffices to consider the operators (2.16) on the intersection T_0 of the cone K_0 with the unit sphere $\|x\| \leqslant 1$; then the elements x_n will belong to K_0.

2.3. Simplicity of a positive characteristic value

2.3.1. *u_0-positive operator with a characteristic vector*

Lemma 2.1. *Let φ_0 be a positive characteristic vector of the u_0-positive operator A.*
Then A is a φ_0-positive operator.

Proof. It follows from the definition of the u_0-positiveness of the operator that for some α_0 and β_0 the inequalities

$$\alpha_0 u_0 \leqslant \varphi_0 \leqslant \beta_0 u_0$$

are satisfied.
Then from the inequalities

$$\alpha u_0 \leqslant A^n x \leqslant \beta u_0$$

it follows that

$$\frac{\alpha}{\beta_0}\varphi_0 \leqslant A^n x \leqslant \frac{\alpha}{\alpha_0}\varphi_0 .$$

This completes the proof of the lemma.

2.3.2. *Simplicity of a characteristic value*

We recall that the characteristic value λ_0 of the linear operator A is called simple if all solutions of the equations

$$(A-\lambda_0 I)^n x = \theta \qquad (n=1, 2, \ldots) \tag{2.26}$$

are, simultaneously, solutions of the equation

$$Ax = \lambda_0 x , \tag{2.27}$$

where the set of solutions of the equation (2.27) is one-dimensional.

Theorem 2.10. *Let $\varphi_0 \in K$ and φ be a characteristic vector of the u_0-positive linear operator A:*

$$A\varphi_0 = \lambda_0 \varphi_0 .$$

Let K be a reproducing cone.
Then λ_0 is a simple characteristic value of the operator A.

Proof. We first show that the set of solutions of the equation (2.27) is one-dimensional. Assuming the contrary, we find that the operator A has a non-zero characteristic vector ψ_0 different from φ_0 (and non-collinear with φ_0), corresponding to the same characteristic value λ_0:

$$A\psi_0=\lambda_0\psi_0 .$$

Without loss of generality, we can assume that $-\psi_0\bar{\in}K$. Then, by virtue of Lemma 1.3, a greatest t_0 can be found such that $\varphi_0-t_0\psi_0\in K$ and $\varphi_0-t\psi_0\bar{\in}K$ for $t>t_0$. This number t_0 is positive. In fact, since the cone K is reproducing, then $\psi_0=\psi_0'-\psi_0''$ where ψ_0', $\psi_0''\in K$. From this it follows that $\psi_0\leqslant\psi_0'$. By virtue of Lemma 2.1, the operator A is φ_0-positive. Therefore, for some p the inequality $A^p\psi_0\leqslant\beta\varphi_0(\beta>0)$ is satisfied, from which it follows that $A^p\psi_0\leqslant\beta\varphi_0$, i.e. $\varphi_0-(\lambda_0^p/\beta)\psi_0\in K$.

Since the operator A is φ_0-positive, then for some n the inequality

$$A^n(\varphi_0-t_0\psi_0)\geqslant\alpha\varphi_0\geqslant\alpha t_0\psi_0$$

will be satisfied where $\alpha>0$. It follows from the last inequality that

$$\varphi_0-t_0\left(1+\frac{\alpha}{\lambda_0^n}\right)\psi_0\in K ,$$

and this contradicts the maximality of t_0.

Thus, a unique (to within the norm) characteristic vector φ_0 of the operator A corresponds to the characteristic value λ_0.

We assume that one of the equations (2.26) has a non-trivial solution χ^* which is not a solution of the equation (2.27). Let $(A-\lambda_0 I)^{n_0}\chi^*=\theta$ and $(A-\lambda_0 I)^{n_0-1}\chi^*\neq\theta$. The element $(A-\lambda_0 I)^{n_0-1}\chi^*$, obviously, will be a solution of the equation (2.27). We have shown that

$$(A-\lambda_0 I)^{n_0-1}\chi^*=k\varphi_0 ,$$

where $k\neq 0$. We introduce the notation

$$-\frac{1}{k}(A-\lambda_0 I)^{n_0-2}\chi^*=\chi_0 .$$

Then

$$A\chi_0=\lambda_0\chi_0-\varphi_0 . \tag{2.28}$$

It follows from (2.28) that $\chi_0\bar{\in}K$ since otherwise the elements $A^n\chi_0=\lambda_0^n\chi_0-n\lambda_0^{n-1}\varphi_0$ will be positive; that is, the elements $(\lambda_0/n)\chi_0-\varphi_0$ will belong to K together with its limit $-\varphi_0$, and this contradicts the definition of the cone. Since the cone K is reproducing, then $\chi_0=\chi_0'-\chi_0''$, where $\chi_0'\in K$ and $\chi_0''\bar{\in}K$, where $\chi_0''\neq\theta$ since $\chi_0\bar{\in}K$. It follows from the φ_0-positiveness of the operator A that $A^p\chi_0''\leqslant\beta\varphi_0(\beta>0)$ for some p. Then

$$\lambda_0^p\chi_0-p\lambda_0^{p-1}\varphi_0=A^p\chi_0=A^p\chi_0'-A^p\chi_0''\geqslant -A^p\chi_0''\geqslant-\beta\varphi_0 ,$$

from which it follows that

$$\chi_0 \geqslant \frac{p\lambda_0^{p-1}-\beta}{\lambda_0^p}\varphi_0\,, \tag{2.29}$$

where

$$p\lambda_0^{p-1}-\beta<0\,,$$

since $\chi_0 \bar{\in} K$.

We denote by t_0 a number such that $\varphi_0+t_0\chi_0 \in K$ and $\varphi_0+t\chi_0 \bar{\in} K$ for $t>t_0$. By virtue of (2.29), the number t_0 is positive. The operator A is positive; therefore, $A(\varphi_0+t_0\chi_0)\in K$, that is, $[1-(t_0/\lambda_0)]\varphi+t_0\chi_0\in K$, and this contradicts the maximality of t_0.

Thus, the equations (2.26) cannot have solutions different from the solutions of equations (2.27).

This completes the proof of the theorem.

2.3.3. *Uniqueness of a positive characteristic vector*

Theorem 2.11. *Let the conditions of Theorem* 2.10 *be satisfied. Then, for the operator* A, *the cone* K *has a unique (to within the norm) characteristic vector.*

Proof. Assuming the contrary, we let

$$A\varphi_1=\lambda_1\varphi_1\,,\quad A\varphi_2=\lambda_2\varphi_2 \qquad (\varphi_1,\varphi_2\in K,\ \|\varphi_1\|=\|\varphi_2\|=1)\,.$$

By virtue of Theorem 2.10, we can assume without loss of generality that $\lambda_1>\lambda_2$. We consider the elements $\varphi(t)$ of the form

$$\varphi(t)=\varphi_2-t\varphi_1 \qquad (0\leqslant t<\infty)\,.$$

We denote by t_0 the value of t such that $\varphi(t_0)\in K$, $\varphi(t)\bar{\in} K$ for $t>t_0$. The number t_0 is positive since by virtue of Lemma 2.1, the element $\varphi(t)$ belongs to K for sufficiently small positive t.

It follows from the positiveness of the operator A that

$$A(\varphi_2-t_0\varphi_1)=\lambda_2\left(\varphi_2-t_0\frac{\lambda_1}{\lambda_2}\varphi_1\right)\in K\,.$$

The inequality $\lambda_1\leqslant\lambda_2$ follows from this and from the maximality of t_0. We have thus arrived at a contradiction.

This completes the proof of the theorem.

2.3.4. *Disposition of invariant subspaces of a positive operator*

The subspace $E_1\subset E$ is called an *invariant subspace* of the linear operator A if $AE_1\subset E_1$.

The collection of all characteristic vectors, completed by adjoining θ,

corresponding to one fixed characteristic value can serve as the simplest example of an invariant subspace of the operator A.

It is not difficult to see that the direct sum of several invariant subspaces will be an invariant subspace. The closure of the direct sum of an arbitrary infinite number of invariant subspaces will also be an invariant subspace. The possibility of the separation of their invariant subspaces is one of the fundamental facts of the theory of linear completely continuous operators. In particular, if φ_0 is a characteristic vector of a linear completely continuous operator A corresponding to a simple characteristic value, then an invariant subspace E_1 exists for the operator A such that every element $x \in E$ is uniquely representable in the form

$$x = f(x)\varphi_0 + x_1, \tag{2.30}$$

where $x_1 \in E_1$ and $f(x)$ is a linear functional. It is obvious that $f(\varphi_0) = 1$ and $f(x) = 0$ for $x \in E_1$.

The following theorem is a strengthening of Theorem 2.11.

Theorem 2.12. *Let the conditions of Theorem* 2.10 *be satisfied. Let Π be an invariant subspace of the operator A to which a positive characteristic vector φ does not belong.*

Let any one of the following three conditions be satisfied:

2.12(a). *The subspace Π is finite-dimensional.*

2.12(b). *The operator A is completely continuous.*

2.12(c). *The space E is weakly complete, the unit sphere in the space E is weakly compact, the cone K allows plastering.*

Then the intersection of the subspace Π with the cone K consists of the single point θ.

Proof. Assuming the contrary, we denote by K_Π the intersection $K \cap \Pi$. Since K and Π are invariant for the operator A, then A transforms K_Π into itself. The set K_Π is a cone.

We shall show that each of the conditions 2.12(a,b,c) suffices for the existence in K_Π of a characteristic vector of the operator A. With this the proof will be completed since the existence of a positive characteristic vector different from φ_0 contradicts Theorem 2.11.

If condition 2.12(a) or 2.12(c) is satisfied, then the existence of a characteristic vector in K_Π follows from Theorem 2.7. We observe that condition 2.12(c) can be weakened assuming that the subspace Π is weakly complete, that the unit sphere in Π is weakly compact, and that the cone K_Π allows plastering.

Let condition 2.12(b) be satisfied. We shall show that the operator A, if it is considered on Π, satisfies the conditions of Theorem 2.5.

Let ψ_0 be a non-zero element of K_Π. By virtue of Lemma 2.1, a natural number p and positive number β can be found such that

$$A^p\psi_0 \leqslant \beta\varphi_0 . \tag{2.31}$$

We introduce the notation $\psi_1 = A^p\psi_0$. By virtue of the same Lemma 2.1, a natural number p_1 and positive number α_1 can be found such that

$$\alpha_1\varphi_0 \leqslant A^{p_1}\psi_1 . \tag{2.32}$$

It follows from (2.31) and (2.32) that

$$A^{p_1}\psi_1 \geqslant \frac{\alpha_1}{\beta} A^p\psi_0 = \frac{\alpha_1}{\beta}\psi_1 .$$

The last relation is condition (2.13) of Theorem 2.5.

This completes the proof of the theorem.

2.3.5. *Integral operators*

The theorems proved in this section are directly applicable to the study of integral operators

$$Ax(t) = \int_\Omega k(t, s)x(s)\,ds . \tag{2.1}$$

Conditions sufficient for the u_0-positiveness can be written in the form

$$u_0(t) \leqslant k_N(t, s) \leqslant \varrho u_0(t) \qquad (t, s \in \Omega) , \tag{2.33}$$

where $k_N(t, s)$ is an iterated kernel.

We make one further observation. If in the proof of the existence of a positive characteristic vector it was appropriate to search out the most "narrow" cone to which the characteristic vector belonged, then for the application of assertions of the type of Theorem 2.12 it is natural to search out the most "broad" cone K for which these assertions remain valid.

2.4. Comparison with other characteristic values

2.4.1. *Fundamental theorem*

It was remarked above (see p. 75) that one of the positive characteristic values of a linear positive completely continuous operator A is the upper bound for the absolute magnitudes of the remaining characteristic values. This fact allows a strengthening for important classes of linear operators

(even for some operators not possessing the property of complete continuity).

Theorem 2.13. *Let the conditions of Theorem* 2.10 *be satisfied. Then a positive characteristic value of the operator* A *(that is, a characteristic value corresponding to a positive characteristic vector* $\varphi \in K$*) is greater than the absolute magnitudes of the remaining characteristic values.*

Proof. We observe first that all positive characteristic values of the operator A are less than λ_0. In fact, let $A\varphi_1 = \lambda_1\varphi_1$, where $\lambda_1 > 0$. By virtue of Theorem 2.10, $\lambda_1 \neq \lambda_0$ and, by virtue of Theorem 2.11, $\varphi_1 \in K$. Therefore, for sufficiently large t the elements $\varphi_0 + t\varphi_1$ do not belong to K. At the same time, for sufficiently small positive t, $\varphi_0 + t\varphi_1 \in K$ (since $-\varphi_1 = \varphi_1' - \varphi_1''$, where $\varphi_1', \varphi_1'' \in K$ and $\varphi_1' \neq \theta$, where for some natural number p the inequality $A^p\varphi_1 < \beta\varphi_0$, $\beta_0 > 0$ is satisfied. It follows from this that

$$-\varphi_1 = \frac{1}{\lambda_1^p} A^p(-\varphi_1) \leqslant \frac{1}{\lambda_1^p} A^p\varphi_1' \leqslant \frac{\beta}{\lambda_1^p}\varphi_0$$

and, finally, $\varphi + (\lambda_1^p/\beta)\varphi_1 \in K$). Thus, the greatest positive number t_0 exists such that $\varphi_0 + t_0\varphi_1 \in K$ and $\varphi_0 + t\varphi_1 \bar{\in} K$ for $t > t_0$. Then $A(\varphi_0 + t_0\varphi_1) \in K$ from which it follows that

$$\lambda_0\left(\varphi_0 + t_0 \frac{\lambda_1}{\lambda_0}\varphi_1\right) \in K \tag{2.34}$$

and (by virtue of the maximality of t_0) $\lambda_1 \leqslant \lambda_0$. This means that $\lambda_1 < \lambda_0$. We assume that the operator A has a non-negative characteristic value λ_1. Then the operator A^2 will have the positive characteristic values λ_0^2 and λ_1^2. The operator A evidently* satisfies the conditions of Theorem 2.10. By what has already been proved, the inequality $\lambda_1^2 < \lambda_0^2$, that is, $|\lambda_1| < \lambda_0$ is satisfied.

It remains for us to consider the case of a non-real characteristic value $\lambda_1 = \sigma + i\tau$. The equality

$$A(x + iy) = (\sigma + i\tau)(x + iy)$$

* If φ_0 is a characteristic vector of the operator A and if A is φ_0-bounded above, then all the operators A^q will be φ_0-bounded above since the inequality

$$A^{mq}x = (A^q)^m x \leqslant \beta A^{m(q-1)}\varphi_0 = \beta\lambda_0^{m(q-1)}\varphi_0$$

follows from $A^m x \leqslant \beta\varphi_0$. Analogously, the φ_0-boundedness below or φ_0-positiveness of the operators A^q follows, respectively, from the φ_0-boundedness below or φ_0-positiveness of the operator A.

is equivalent to a system of two equalities:

$$\begin{aligned} Ax &= \sigma x - \tau y\,, \\ Ay &= \tau x + \sigma y\,. \end{aligned} \tag{2.35}$$

Thus, we must show that the inequality

$$|\lambda_1| = \sqrt{\sigma^2+\tau^2} < \lambda_0$$

follows from the equalities (2.35) where $\|x\| + \|y\| > 0$. We observe that it follows from the equalities (2.35) and the non-reality of λ_1 that x and y are linearly independent and, in particular, different from zero.

We denote by Π the two-dimensional plane in the space E consisting of elements of the form $\xi x + \eta y$. It follows from (2.35), on the one hand, that $\varphi_0 \in \Pi$ and, on the other, that Π is an invariant two-dimensional subspace of the operator. By virtue of Theorem 2.12, the intersection $\Pi \cap K$ consists of the single point θ. This means that the vectors $\xi x + \eta y$ for $|\xi| + |\eta| > 0$ do not belong to K.

Since the cone K is reproducing, then $x = x' - x''$ where $x', x'' \in K$, and where $x'' \neq \theta$, because we have proved that $x \bar{\in} K$. It follows from the φ-positiveness of the operator A that $A^p x'' \leqslant \beta\varphi_0 (\beta > 0)$ for some p. Then

$$A^p x = A^p x' - A^p x'' \geqslant -A^p x'' \geqslant -\beta\varphi_0\,,$$

that is

$$\varphi_0 + \frac{1}{\beta} A^p x \in K\,.$$

It follows from the equalities (2.35) that the element $\psi = \varphi_0 + (1/\beta)A^p x$ can be written in the form

$$\psi = \xi x + \eta y + \varphi_0 \qquad (|\xi| + |\eta| > 0)\,. \tag{2.36}$$

We denote by T the collection of all elements of the form (2.36) lying in K. We have shown that such elements exist.

To each $\psi \in T$ we assign the number $\xi^2 + \eta^2$. The resulting functional will be bounded. It follows from the closure of the cone that it will take on at some points its greatest value M. Let $\psi_0 = \xi_0 x + \eta_0 y + \varphi_0$ be one of these points.

It follows from the φ_0-positiveness of the operator A that

$$A^p \psi_0 \geqslant \alpha\varphi_0 \qquad (\alpha > 0) \tag{2.37}$$

for some p. Without loss of generality, we can assume that $\alpha/\lambda_0^p < 1$. The inequality (2.37) can be rewritten in the form

$$\left(1 - \frac{\alpha}{\lambda_0{}^p}\right)\lambda_0{}^p\varphi_0 + (\xi_1 x + \eta_1 y) \geqslant \theta\,, \tag{2.38}$$

where

$$\xi_1 x + \eta_1 y = A^p(\xi_0 x + \eta_0 y)\,.$$

Direct calculation shows that

$$\xi_1{}^2 + \eta_1{}^2 = (\sigma^2 + \tau^2)^p(\xi_0{}^2 + \eta_0{}^2) = M|\lambda_1|^{2p}\,.$$

It is obvious that

$$\psi_1 = \varphi_0 + \frac{\xi_1}{\left(1 - \frac{\alpha}{\lambda_0{}^p}\right)\lambda_0{}^p}\,x + \frac{\eta_1}{\left(1 - \frac{\alpha}{\lambda_0{}^p}\right)\lambda_0{}^p}\,y$$

is an element of the form (2.36). Therefore

$$\left(\frac{\xi_1}{\lambda_0{}^p - \alpha}\right)^2 + \left(\frac{\eta_1}{\lambda_0{}^p - \alpha}\right)^2 = \frac{M|\lambda_1|^{2p}}{(\lambda_0{}^p - \alpha)^2} \leqslant M\,,$$

from which it follows that $|\lambda_1| < \lambda_0$.
This completes the proof of the theorem.

2.4.2. *Characteristic values of operators which are bounded above*
The theorems proved above for u_0-positive operators lose their validity if we go over to arbitrary positive operators. The identity operator can serve as an example. However, some of the proven theorems in weakened form can be transferred to wider classes of operators.
We restrict ourselves in this subsection to one theorem for the operators A which are u_0-bounded above ($u_0 \in K$, $u_0 \neq \theta$), that is, for positive operators such that for every non-zero $x \in K$ an m and a $\beta > 0$ can be found such that $A^m x \leqslant \beta u_0$. We recall that every positive linear operator is u_0-bounded above if the cone K is solid and u_0 is an interior element of K.

Theorem 2.14. *Let h_0 be a positive characteristic vector of the linear operator A which is h_0-bounded above*:

$$Ah_0 = \lambda_0 h_0\,.$$

Let the cone K be reproducing.
Then the remaining characteristic values of the operator A are in modulus not greater than λ_0.

Proof. Let $Ag = \lambda g (-g \overline{\in} K)$. We first consider the case when $\lambda > 0$.
Since the cone K is reproducing, then $g = g' - g''$, where $g', g'' \in K$. Let $A^m g' \leqslant \beta h_0$; then

$$A^m g = A^m g' - A^m g'' \leqslant A^m g' \leqslant \beta h_0 .$$

Therefore

$$g = \frac{1}{\lambda^m} A^m g \leqslant \frac{\beta}{\lambda^m} h_0 .$$

We denote by t_0 a positive number such that $h_0 - t_0 g \in K$ and $h_0 - tg \bar{\in} K$ for $t > t_0$. Then it follows from the relations

$$A(h_0 - t_0 g) = \lambda_0 \left(h_0 - \frac{\lambda t_0}{\lambda_0} \right) \in K$$

that $\lambda \leqslant \lambda_0$.

Now let $Ag = \lambda g (-g \bar{\in} K)$ and $\lambda < 0$. We then consider the operator $\boldsymbol{B} = \boldsymbol{A}^2$. The operator $\boldsymbol{B}$ will also be h_0-bounded above;* in this connection, λ^2 will be a positive characteristic value of the operator $\boldsymbol{B}$. By what has already been proved, $\lambda^2 \leqslant \lambda_0^2$. Thus, $|\lambda| \leqslant \lambda_0$.

It remains to consider the case of complex characteristic values. Here we shall make use of considerations which are applicable to the second half of the proof of Theorem 2.13.

The equality $A(x+iy) = \lambda(x+iy)$ where $\lambda = \sigma + i\tau$ is equivalent to the system of two equalities

$$\begin{aligned} Ax &= \sigma x - \tau y , \\ Ay &= \tau x + \sigma y . \end{aligned} \tag{2.35}$$

Therefore, we must show that the inequality $\sigma^2 + \tau^2 \leqslant \lambda_0^2$ follows from these equalities.

As in the proof of Theorem 2.13, we consider the set of elements of the form

$$\psi = \xi x + \eta y + h_0 \qquad (|\xi| + |\eta| > 0) ,$$

which has a non-empty intersection T with the cone K.

T will be a compact set not containing the point θ. The proof of all these statements under the conditions of Theorem 2.14 almost completely repeats the corresponding part of the proof of Theorem 2.13.**

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* See footnote on p. 81.

** Since it is impossible to refer to Theorem 2.2, this requires a modified proof of the fact that the plane Π in which the operator A is defined by the equalities (2.35) with a non-zero τ does not have non-zero elements in common with the cone K. The indicated fact is valid for arbitrary positive linear operators A.

In order to verify this, we shall consider the plane Π as Euclidean and assume the numbers $\{\xi, \eta\}$ to be rectangular coordinates of the point $\xi x + \eta y \in \Pi$. It then follows

We denote by $\psi_0=\xi_0 x+\eta_0 y+h_0$ an element of T such that for all remaining elements $\xi x+\eta y+h_0\in T$ the inequality

$$\xi^2+\eta^2\leqslant\xi_0^2+\eta_0^2$$

is satisfied. It is obvious that $(1/\lambda_0)A\psi_0\in T$, that is,

$$\frac{\sigma\xi_0+\tau\eta_0}{\lambda_0}x+\frac{\sigma\eta_0-\tau\xi_0}{\lambda_0}y+h_0\in T.$$

Therefore

$$\left(\frac{\sigma\xi_0+\tau\eta_0}{\lambda_0}\right)^2+\left(\frac{\sigma\eta_0-\tau\xi_0}{\lambda_0}\right)^2\leqslant\xi_0^2+\eta_0^2,$$

that is

$$\sigma^2+\tau^2\leqslant\lambda_0^2.$$

This completes the proof of the theorem.

2.4.3. *Positive characteristic vectors of operators which are bounded below* In this subsection, we shall assume that A is a u_0-bounded below operator where u_0 is a positive vector of the operator A:

$$Au_0=\lambda_0 u_0.$$

In this case, the operator A can have positive characteristic values greater than λ_0. As an example, the operator

$$A\{\xi,\eta\}=\{2\xi,\xi+\eta\}$$

can be considered in a two-dimensional space of vectors $\{\xi,\eta\}$ with the cone K of elements for which $\xi,\eta\geqslant 0$; this operator is u_0-bounded below where $u_0=\{0,1\}$ and $Au_0=u_0$. At the same time, the operator A has a characteristic vector $\varphi_0=\{1,1\}$ in the cone K where $A\varphi_0=2\varphi_0$.
However, some properties of the u_0-positive operators are retained in the

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from the formulas (2.35) that the operator

$$A_1=\frac{1}{\sqrt{(\sigma^2+\tau^2)}}A$$

defines in the plane Π a rotation through some angle which is not zero and different from π. Let the non-zero vector $\varphi_0=\{\xi_0,\eta_0\}$ belong to Π and K; then all the vectors $A_1^n\varphi_0$ $(n=1,2,\ldots)$ belong to Π and K together with the convex hull which evidently contains the zero element θ together with some neighborhood in Π. It follows from this that some non-zero element x belongs to the cone K together with the element $-x$. We have thus arrived at a contradiction.

transition to operators which are u_0-bounded below. We single out one of them:

Theorem 2.15. *Let*

$$Ah_0 = \lambda_0 h_0 \qquad (h_0 \in K,\ \|h_0\| = 1),$$

and let A be an operator which is h_0-bounded below.
Then a characteristic value strictly greater than λ_0 corresponds to each normed positive characteristic vector φ_0, different from h_0, of the operator A (if such characteristic vectors exist).

Proof. Let $A\varphi_0 = \lambda\varphi_0$, $\varphi_0 \in K$ and $\|\varphi_0\| = 1$. It follows from the lower h_0-boundedness of the operator A that

$$A^p\varphi_0 \geqslant \alpha h_0,$$

where p is some natural number and $\alpha > 0$. Then $\varphi_0 \geqslant \gamma_0 h_0$ where $\gamma_0 = \alpha/\lambda^p$.
It follows from the last inequality that $A\varphi_0 \geqslant \gamma_0 Ah_0$, that is $\lambda\varphi_0 \geqslant \gamma_0\lambda_0 h_0$. Applying the operator A to this inequality, we obtain $\lambda^2\varphi_0 \geqslant \gamma_0\lambda_0^2\lambda_0$ and analogously

$$\varphi_0 \geqslant \gamma_0\left(\frac{\lambda_0}{\lambda}\right)^n h_0 \qquad (n = 1, 2, \ldots).$$

Thus, $\lambda_0 \leqslant \lambda$.
We assume that $\lambda = \lambda_0$. We denote by t_0 a number such that $\varphi_0 \geqslant t_0 h_0$ and $\varphi_0 - th_0 \in K$ for $t > t_0$. By what has already been proved, $t_0 > 0$. It follows from the lower h_0-boundedness of the operator A that $A^q(\varphi_0 - t_0 h_0) \geqslant \alpha_0 h_0$ where $\alpha_0 > 0$ if $\varphi_0 \neq t_0 h_0$. Then

$$\lambda_0^q(\varphi_0 - t_0 h_0) \geqslant \alpha_0 h_0,$$

that is

$$\varphi_0 - \left(t_0 + \frac{\alpha_0}{\lambda_0^q}\right) h_0 \in K,$$

which contradicts the maximality of t_0.
This completes the proof of the theorem.

2.5. Non-homogeneous linear equations

2.5.1. *Formulation of the problem*
We pass to the study of non-homogeneous equations

$$\lambda\varphi = A\varphi + f, \tag{2.39}$$

where A is a linear positive operator.

We assume that λ_0 is the least upper bound of the absolute magnitudes of the characteristic values of the operator A. Then equation (2.39) has a unique solution for an arbitrary f if $\lambda > \lambda_0$. If $\lambda = \lambda_0$ or $\lambda < \lambda_0$, then the question about the solvability of the equation (2.39) becomes more complicated: this equation is solvable for all $f \in E$, if λ does not belong to the spectrum of the operator and only for some f otherwise.

A simpler and more complete answer to the question of the solvability of a non-homogeneous equation (2.39) can be obtained if we consider elements f only from the cone K and if we seek solutions in the cone K.

Theorem 2.16. *If $f \in K$ and $\lambda > \lambda_0$, then the equation* (2.39) *has a unique solution in the cone K.*

If $\lambda \leqslant \lambda_0$ and

$$A^n f \geqslant \varepsilon u_0 \qquad (\varepsilon > 0), \tag{2.40}$$

where u_0 is a non-zero element of K such that

$$A^p u_0 \geqslant \lambda_0^p u_0, \tag{2.41}$$

then the equation (2.39) *does not have solutions in K.*

2.5.2. *Lemmas about equivalent norms*

Let A be a linear operator acting in a Banach space E. We denote by ϱ_0 the smallest radius of the circle $|\lambda| \leqslant \varrho$ in the complex plane (λ) in which the spectrum of the operator A lies.

It is obvious that $\|A\| \geqslant \varrho_0$, where $\|A\| = \varrho_0$ only in exceptional cases. For example, the Volterra integral operator

$$Ax(t) = \int_0^t k(t, s)x(s)ds$$

with a continuous kernel does not have non-zero characteristic values ($\varrho_0 = 0$); however, its norm is different from zero no matter in what space of functions (C or L_p) it is considered.

In the theory of linear operators the important equality*

$$\lim_{n\to\infty} \sqrt[n]{\|A^n\|} = \varrho_0 \tag{2.42}$$

* See, for example, I. M. Gelfand, D. A. Raikov and G. E. Šilov [1].

is proved. Using this equality, we can construct a norm $\|x\|_0$ in the space E which is equivalent to the one originally defined in which the operator A has the norm

$$\|A\|_0 = \sup_{\|x\|_0 \leqslant 1} \|Ax\|_0 ,$$

not exceeding $\varrho_0 + \varepsilon$, where ε is an arbitrary positive number (defined by the construction of the new norm).

We recall that the norm $\|x\|_0$ is called *equivalent* to the norm $\|x\|$ if

$$m\|x\| \leqslant \|x\|_0 \leqslant M\|x\| \qquad (x \in E) ,$$

where m, M are fixed positive numbers. It is not difficult to see that the continuity (or complete continuity) of an operator acting in the space E does not depend on which one of equivalent norms is used.

Let $\varepsilon_0 > 0$ be given. By virtue of (2.42), an n_0 can be found such that

$$\|A^{n_0}x\| \leqslant (\varrho_0 + \varepsilon_0)^{n_0}\|x\| \qquad (x \in E) . \tag{2.43}$$

We set

$$\|x\|_0 = \|x\| + \frac{\|Ax\|}{\varrho_0 + \varepsilon_0} + \frac{\|A^2x\|}{(\varrho_0 + \varepsilon_0)^2} + \ldots + \frac{\|A^{n_0-1}x\|}{(\varrho_0 + \varepsilon_0)^{n_0-1}} \quad (x \in E) . \tag{2.44}$$

The norm axioms for $\|x\|_0$ are verified without difficulty; the equivalence of the norm (2.55) to the original norm follows from the inequalities

$$\|x\| \leqslant \|x\|_0 \leqslant \left[1 + \frac{\|A\|}{\varrho_0 + \varepsilon_0} + \ldots + \frac{\|A\|^{n_0-1}}{(\varrho_0 + \varepsilon_0)^{n_0-1}}\right] \|x\| \quad (x \in E) . \tag{2.45}$$

Lemma 2.2. *Let the norm* (2.44) *be considered in the space E. Then*

$$\|A\|_0 \leqslant \varrho_0 + \varepsilon_0 . \tag{2.46}$$

Proof. It follows from (2.44) that

$$\|Ax\|_0 = (\varrho_0 + \varepsilon_0)\|x\|_0 + \frac{\|A^{n_0}x\|}{(\varrho_0 + \varepsilon_0)^{n_0-1}} - (\varrho_0 + \varepsilon_0)\|x\| ,$$

from which by virtue of (2.43)

$$\|Ax\|_0 \leqslant (\varrho_0 + \varepsilon_0)\|x\|_0 \qquad (x \in E) .$$

This completes the proof of the lemma.

In the transition to the equivalent norm some properties of the original norm can be lost. For example, let the original norm be monotonic with

respect to the partial ordering which is generated by the cone K. It follows from $x \leqslant y (x, y \in K)$ that $\|x\| \leqslant \|y\|$. The equivalent norm $\|x\|_0$ cannot be monotonic. The following lemma holds.

Lemma 2.3. *Let $\|x\|$ be monotonic and let the linear operator A leave K invariant.*
Then the norm (2.44) *is also monotonic.*

Proof. The operators $A, A^2, \ldots, A^{n_0-1}$ are monotonic. Therefore, it follows from $x \leqslant y$ that $A^i x \leqslant A^i y$ $(i=1, \ldots, n_0-1)$ and by virtue of the monotonicity of the original norm $\|A^i x\| \leqslant \|A^i y\|$ $(i=1, \ldots, n_0-1)$.
But then for $\theta \leqslant x \leqslant y$

$$\|x\|_0 = \|x\| + \frac{\|Ax\|}{\varrho_0+\varepsilon_0} + \ldots + \frac{\|A^{n_0-1}x\|}{(\varrho_0+\varepsilon_0)^{n_0-1}} \leqslant$$

$$\leqslant \|y\| + \frac{\|Ay\|}{\varrho_0+\varepsilon_0} + \ldots + \frac{\|A^{n_0-1}y\|}{(\varrho_0+\varepsilon_0)^{n_0-1}} = \|y\|_0 .$$

This completes the proof of the lemma.

2.5.3. *Proof of Theorem 2.16*
Let $f \in K$ and $\lambda > \lambda_0$ where λ_0 is the radius of the circle in which the spectrum of the operator A is situated. We choose a number $q<1$ such that

$$\lambda_0 < q\lambda < \lambda .$$

By virtue of Lemma 2.2, a norm $\|x\|_0$ can be introduced in E such that

$$\|A\|_0 = \sup_{\|x\|_0 \leqslant 1} \|Ax\|_0 < q\lambda . \tag{2.47}$$

We consider the series

$$R_\lambda f = \sum_{k=0}^{\infty} \frac{A^k f}{\lambda^{k+1}} . \tag{2.48}$$

This series converges in the new norm since

$$\left\| \frac{A^k f}{\lambda^{k+1}} \right\| \leqslant \frac{\|A^k\|_0}{\lambda^{k+1}} \|f\|_0 \leqslant q^k \frac{\|f\|_0}{\lambda} \qquad (k=1, 2, \ldots) ,$$

and, consequently, in the usual norm also. The sum $R_\lambda f$ of the series (2.48) is a solution of the equation (2.39) since

$$AR_\lambda f = \sum_{k=0}^{\infty} \frac{A^{k+1} f}{\lambda^{k+1}} = \lambda \sum_{k=1}^{\infty} \frac{A^k f}{\lambda^{k+1}} = \lambda R_\lambda f - f .$$

If $f \in K$, then all the terms of the series (2.48) belong to K and together with them the element $R_\lambda f$ belongs to K.

The uniqueness of the solution of the equation (2.39) under the condition $\lambda > \lambda_0$ is evident. Thus, the first part of the statements of Theorem 2.16 is proved.

We pass to the case when $\lambda \leqslant \lambda_0$ and when the inequalities (2.40) and (2.41) are satisfied.

For $\lambda \leqslant 0$ the statement of the theorem is obvious. In the case when $0 < \lambda \leqslant \lambda_0$, we shall prove the theorem by contradiction.

Let

$$A\varphi^* + f = \lambda\varphi^* ,$$

where $\varphi^* \in K$ and $\lambda \leqslant \lambda_0$. From this follows the equality

$$A^k\varphi^* + A^{k-1}f + \lambda A^{k-2}f + \ldots + \lambda^{k-1}f = \lambda^k\varphi^* , \tag{2.49}$$

where k is an arbitrary natural number.

It follows from (2.49) (for $k = n+1$) and (2.40) that

$$\varphi^* \geqslant \frac{1}{\lambda^{n+1}} A^n f \geqslant \frac{\varepsilon}{\lambda^{n+1}} u_0 .$$

Therefore, a positive number t_0 exists such that $\varphi^* - t_0 u_0 \in K$ and $\varphi^* - tu_0 \overline{\in} K$ for $t > t_0$.

From the equality (2.49) in which $k = p(n+1)$ we subtract the inequality

$$t_0 A^{p(n+1)} u_0 \geqslant t_0 \lambda^{p(n+1)} u_0 ,$$

which follows from (2.41). After the subtraction, we obtain the relation

$$A^{p(n+1)}(\varphi^* - t_0 u_0) + A^{p(n+1)-1}f + \lambda A^{p(n+1)-2}f + \ldots$$
$$\ldots + \lambda^{p(n+1)-1}f \leqslant \lambda^{p(n+1)}(\varphi^* - t_0 u_0).$$

The inequality

$$\lambda^{(p-1)(n+1)} A^n f \leqslant \lambda^{p(n+1)}(\varphi^* - t_0 u_0)$$

follows from this since $n \leqslant p(n+1) - 1$, and from this inequality, by virtue of (2.40),

$$\varphi^* - t_0 u_0 \geqslant \frac{\varepsilon}{\lambda^{n+1}} u_0 .$$

The last inequality contradicts the maximality of t_0.

This completes the proof of the theorem.

It follows, in particular, from this theorem that the equation (2.39) with the u_0-positive operator A does not have solutions in the cone K for

$f \in K (f \neq \theta)$ and for $\lambda \leqslant \lambda_0$ where λ_0 is a positive characteristic value.
The second part of the statement of Theorem 2.16 (as well as a number of other propositions of the preceding sections) can be considered as propositions about the incompatibility of some of the partial ordering relations. Corresponding theorems are stated in the following subsection.

2.5.4. *Incompatible inequalities*
In the sequel, we shall write

$$x \overline{\leqslant} y ,$$

if $y - x \overline{\in} K$.

Theorem 2.17. *Let the operator A be u_0-bounded below where*

$$Au_0 \geqslant \lambda_0 u_0 . \tag{2.50}$$

Then for an arbitrary non-zero $x \in K$, for $\lambda < \lambda_0$

$$Ax \overline{\leqslant} \lambda x , \tag{2.51}$$

and it follows from $Ax \leqslant \lambda_0 x$ $(x \in K)$ that

$$x = ku_0 \quad \text{and} \quad Au_0 = \lambda_0 u_0 .$$

Proof. We first show the validity of the relation (2.51).
Let $Ax \leqslant \lambda x$. It follows from the lower u_0-boundedness of the operator A that

$$A^p x \geqslant \alpha u_0 \qquad (\alpha > 0) \tag{2.52}$$

for some p. We set $A^p x = x_0$; then $x_0 \geqslant \alpha u_0$ and

$$Ax_0 \leqslant \lambda x_0 . \tag{2.53}$$

By virtue of (2.50) and (2.53),

$$\lambda x_0 \geqslant Ax_0 \geqslant \alpha Au_0 \geqslant \alpha \lambda_0 u_0 .$$

By repeating this argument n times, we obtain the inequality

$$\lambda^n x_0 \geqslant \alpha \lambda_0^n u_0 ,$$

that is,

$$x_0 \geqslant \alpha \left(\frac{\lambda_0}{\lambda} \right)^n u_0 \qquad (n = 1, 2, \ldots) .$$

Therefore

$$\lambda_0 \leqslant \lambda .$$

The relation (2.51) is thus proved.
Now let $Ax \leqslant \lambda_0 x$. Again we find a natural number p such that (2.52) is

satisfied where, in addition, we require that this p be minimal.
We first assume that $x_0 = A^p x \neq k u_0$. Then a $t_0 > 0$ can be found such that $x_0 \geqslant t_0 u_0$ and $x \overline{\geqslant} t u_0$ for $t > t_0$. But by virtue of the lower u_0-boundedness of the operator A for some natural number and positive number α_1

$$A^q(x_0 - t_0 u_0) \geqslant \alpha_1 u_0 ,$$

from which, by virtue of (2.50), it follows that

$$\lambda_0^q x_0 \geqslant A^q x_0 = A^q(x_0 - t_0 u_0) + t_0 A^q u_0 \geqslant (\alpha_1 + \lambda_0^q t_0) u_0 ,$$

which contradicts the maximality of t_0.
Thus, $A^p x = k_0 u_0$. If $p = 0$, then the proof is completed. If $p > 0$, then we introduce the notation

$$y = \lambda_0 A^{p-1} x - k_0 u_0 .$$

It follows from the minimality of p that $y \neq \theta$. Obviously

$$y = A^{p-1}(\lambda_0 x - Ax) \geqslant \theta$$

and, consequently, $Ay \geqslant \theta$. On the other hand,

$$Ay = k_0(\lambda_0 u_0 - Au_0) \leqslant \theta .$$

This means that $Ay = \theta$ and this equality contradicts the lower u_0-boundedness of the operator A.
This completes the proof of the theorem.

Theorem 2.18. *Let the operator A be u_0-bounded above where*

$$Au_0 \leqslant \lambda_0 u_0 . \tag{2.54}$$

Then for an arbitrary non-zero $x \in K$ for $\lambda > \lambda_0$

$$Ax \overline{\geqslant} \lambda x . \tag{2.55}$$

The proof literally repeats the first part of the proof of the preceding theorem.
We note that under the conditions of the Theorem 2.18, elements different from u_0 can exist for which $Ax_0 \geqslant \lambda_0 x_0$. The operator

$$A\{\xi, \eta\} = \{\lambda_0 \xi, \lambda_0 \xi\}$$

in the two-dimensional space of pairs $\{\xi, \eta\}$ with the cone K with non-negative coordinates can serve as an example; this operator is u_0-bounded above, where $u_0 = \{1, 1\}$ and $Au_0 = \lambda_0 u_0$, but for all $x = \{\xi, \eta\}$, where $0 \leqslant \eta \leqslant \xi$, the inequality

$$Ax = \{\lambda_0 \xi, \lambda_0 \xi\} \geqslant \{\lambda_0 \xi, \lambda_0 \eta\} = \lambda_0 x$$

is satisfied.

Theorem 2.19. *Let the operator A be u_0-positive where*

$$Au_0 = \lambda_0 u_0 . \tag{2.56}$$

Then for an arbitrary non-zero $x \in K$ $(x \neq ku_0)$ the elements $\lambda_0 x$ and Ax are incomparable:

$$\lambda_0 x \overline{\leqslant} Ax , \quad \lambda_0 x \overline{\geqslant} Ax_0 .$$

Proof. The relation $\lambda_0 x \overline{\geqslant} Ax$ follows from Theorem 2.17. The relation $\lambda_0 x \overline{\leqslant} Ax$ is proved analogously.
Assuming the contrary, let $\lambda_0 x \leqslant Ax$. We choose a minimal p such that $A^p x \leqslant \beta u_0$. Let $A^p x \leqslant t_0 u_0$ and $A^p x \leqslant t u_0$ for $t < t_0$; the number t_0 is positive since

$$t_0 u_0 \geqslant A^p x \geqslant \lambda_0^p x .$$

We assume that $A^p x \neq ku_0$. Then $t_0 u_0 - A^p x \neq \theta$ and a natural number q can be found such that

$$A^q (t_0 u_0 - A^p x) \geqslant \alpha u_0 ,$$

where $\alpha > 0$. But then

$$(t_0 \lambda_0^q - \alpha) u_0 \geqslant A^{p+q} x \geqslant \lambda_0^q A^p x ,$$

that is,

$$A^p x \leqslant \left(t_0 - \frac{\alpha}{\lambda_0^q} \right) u_0 ,$$

which contradicts the minimality of t_0.
Thus, $A^p x = k_0 u_0$. We set

$$y = \lambda_0 A^{p-1} x - k_0 u_0 .$$

It follows from (2.56) that

$$Ay = \lambda_0 A^p x - k_0 A u_0 = k_0 (\lambda_0 u_0 - A u_0) = \theta ,$$

and this contradicts the lower u_0-boundedness of the operator A.
This completes the proof of the theorem.

The statement of this theorem in other terms was obtained above for completely continuous u_0-positive linear operators. In fact, the elements of the form $Ax - \lambda_0 x$ $(x \in E)$ form an invariant subspace of the completely continuous operator, not containing a characteristic vector u_0 if λ_0 is a simple characteristic value corresponding to the characteristic vector u_0. Therefore, the statement of Theorem 2.19 for u_0-positive completely continuous operators follows from Theorem 2.10 and Theorem 2.12.
An application of the theorems of this subsection is indicated below.

2.5.5. *Comparison of the characteristic values of two operators*

We shall consider various operators which leave the same cone K invariant.

We shall say that the operator A_2 is *greater* than the operator A_1 and write $A_1 \leqslant A_2$ if

$$A_1 x \leqslant A_2 x \qquad (x \in K). \tag{2.57}$$

The described partial ordering in the space of linear continuous operators is the usual partial ordering generated by the cone of the non-negative operators (the non-negative operators form the cone in the space of the operators if the original cone K in the space E possesses the property that the linear hull is dense in the space E). We use the sign $\leqslant$ for the partial ordering relation for the space E as well as for the space of the operators. As an example, we consider the integral operators

$$A_1 x(t) = \int_\Omega k_1(t, s) x(s) ds$$

and

$$A_2 x(t) = \int_\Omega k_2(t, s) x(s) ds$$

acting in some function space E.

If

$$0 \leqslant k_1(t, s) \leqslant k_2(t, s) \qquad (t, s \in \Omega),$$

then both operators leave the cone of non-negative functions invariant where $A_1 \leqslant A_2$.

We assume that the positive linear operators A_1 and A_2 satisfying the condition (2.57), are completely continuous. We further assume that

$$A_1 u_1 = \lambda_1 u_1 \quad (u_1 \in K, \quad u_1 \neq \theta, \quad \lambda_1 > 0).$$

Then

$$A_2 u_1 \geqslant \lambda_1 u_1,$$

and, by virtue of Theorem 2.5, the operator A_2 has a positive characteristic vector u_2: $A_2 u_2 = \lambda_2 u_2$, where $\lambda_2 \geqslant \lambda_1$.

Comparative estimates for the characteristic values of the operators A_1 and A_2 satisfying condition (2.57), without assuming their complete continuity, can be obtained from the theorems of the preceding subsection. Let us assume that the operator A_1 is u_0-bounded below where the condition (2.50) is satisfied,

$$A_1 u_0 \geqslant \lambda_0 u_0 .$$

Let λ_2 be a positive characteristic value of the operator A_2:

$$A_2 \varphi = \lambda_2 \varphi .$$

It then follows from $A_1 \leqslant A_2$ that

$$A_1 \varphi \leqslant \lambda_2 \varphi$$

and, by virtue of Theorem 2.17, $\lambda_0 \leqslant \lambda_2$. Moreover, by virtue of the same theorem, $\lambda_0 < \lambda_2$ if $\varphi \neq k u_0$.

Analogously, let the operator A_2 be u_0-bounded above where the condition (2.54) is satisfied:

$$A_2 u_0 \leqslant \lambda_0 u_0 .$$

Let $A_1 \psi = \lambda_1 \psi$ ($\psi \in K$). Then it follows from $A_1 \leqslant A_2$ that

$$A_2 \psi \leqslant \lambda_1 \psi$$

and, by virtue of Theorem 2.18, $\lambda_1 \leqslant \lambda_0$.

In the statements which have been formulated, it is convenient to choose as u_0 the positive characteristic vector of the corresponding operator.

Chapter 3

DIFFERENTIABILITY WITH RESPECT TO A CONE

In the study of non-linear operators which leave a cone invariant, it is convenient to make use of the special concept of the derivatives. Differentiability with respect to a cone is a less restrictive assumption than differentiability according to Gâteaux, and strong differentiability with respect to a cone is a less restrictive condition than differentiability according to Fréchet.

It is convenient to formulate criteria for the positiveness of a non-linear operator, its monotoneity and so on in terms of the derivative. It is easy to determine with derivatives whether or not there are elements in the cone which satisfy the special relations of a partial ordering.

3.1. Derivatives with respect to a cone

3.1.1. *Gâteaux and Fréchet derivatives*

We recall some definitions.

The function $x(t)$ $(a \leqslant t \leqslant b)$ with values in a Banach space E is called *differentiable at the point* t if there exists an element $x'(t_0) \in E$ such that

$$\lim_{\Delta t \to 0} \frac{x(t_0+\Delta t)-x(t_0)}{\Delta t} = x'(t_0)\,. \tag{3.1}$$

The derivative $x'(t)$ is called *strong* if a strong limit (3.1) (limit in norm) is considered, and *weak* if a weak limit is considered. The properties of the derivative $x'(t)$ are basically analogous to the properties of the derivatives of usual scalar functions. We shall use these properties without mentioning them explicitly.

The non-linear operator A is called *differentiable* at the point x_0 in the direction h if the abstract function

$$y(t)=A(x_0+th) \tag{3.2}$$

is differentiable with respect to t at the point $t=0$. Corresponding to the strongly or weakly differentiable function (3.2), we shall speak about the

strong or *weak differentiability* of the operator A at the point x_0 in the direction h.

We assume that the operator A is differentiable at the point x_0 in all directions $h \in E$. This means that the functions (3.2) are differentiable with respect to t at the point $t=0$ for all $h \in E$. In many cases, the derivative of a function (3.2) has the form

$$y'(0) = A'(x_0)h\,, \tag{3.3}$$

where $A'(x_0)$ is a linear operator. We call the operator $A'(x_0)$ a *weak* or *strong Gâteaux derivative* of the operator A at the point x_0 depending on whether weak or strong derivatives of the functions (3.2) are considered. If the increment $A(x_0+h)-Ax_0$ is representable in the form

$$A(x_0+h)-Ax_0 = A'(x_0)h+\omega(x_0, h) \qquad (h \in E)\,, \tag{3.4}$$

where $A'(x_0)$ is a linear operator and

$$\lim_{\|h\|\to 0} \frac{\omega(x_0, h)}{\|h\|} = \theta\,, \tag{3.5}$$

then the operator $A'(x_0)$ is called a *Fréchet derivative* (*weak* or *strong* depending on whether a strong or weak limit is implied in the equality (3.5)). It is clear that an operator which is differentiable according to Fréchet is differentiable according to Gâteaux and the Gâteaux and Fréchet derivatives coincide.

The operator A is called *differentiable* (*strongly* or *weakly* according to Gâteaux or Fréchet) on the set $G \subset E$ if it is differentiable at every point $x \in G$. If the operator A is differentiable on $G \subset E$, then the derivative $A'(x)$ can be considered as the operator defined on G with values in the space of continuous linear operators on E.

In the consideration of concrete operators, it is, all in all, easier to prove the existence of a weak Gâteaux derivative. More consequences can be derived from the existence of a strong Fréchet derivative. It is useful to keep in mind that *it follows from the existence on some open set of a weak Gâteaux derivative which is continuous in the norm of the linear operators that this weak Gâteaux derivative is simultaneously a strong Fréchet derivative.* The *Riemann integral* of a function with values in a Banach space E is defined in a natural way (as a strong or weak limit of integral sums). In the sequel, the estimate

$$\left\| \int_a^b x(t)\,dt \right\| \leqslant \sup_{a \leqslant t \leqslant b} \|x(t)\|(b-a) \tag{3.6}$$

is often used and the equality

$$\int_a^b y'(t)\,dt = y(b) - y(a) \tag{3.7}$$

is valid, for example, for the continuous functions $y(t)$ with a piecewise continuous derivative $y'(t)$. In particular, it follows from (3.7) that

$$A(x+h) - Ax = \int_0^1 A'(x+th)h\,dt\,, \tag{3.8}$$

if $A'(x+th)h$ is continuous.

3.1.2. *Definitions of derivatives with respect to a cone*

We shall say that the operator A is *differentiable at the point* x_0 *in the directions of the cone* $K \subset E$ if the function $y(t) = A(x_0 + th)$ is differentiable with respect to t at the point $t = 0$ for all $h \in K$. If the derivative $y'(0)$ for $h \in K$ allows the representation (3.3), then the linear operator $A'(x_0)$ is a (*weak* or *strong*) *derivative with respect to the cone* K of the operator A at the point x_0. In this connection, we shall call the operator A (*strongly* or *weakly*) *differentiable in the cone.*

If

$$A(x_0+h) - Ax_0 = A'(x_0)h + \omega(x_0, h) \qquad (h \in K)\,, \tag{3.9}$$

where $A'(x_0)$ is a linear operator and

$$\lim_{h \in K,\, \|h\| \to 0} \frac{\omega(x_0, h)}{\|h\|} = \theta\,, \tag{3.10}$$

then we shall call $A'(x_0)$ the (*strong* or *weak*) *Fréchet derivative with respect to the cone* K.

If the Fréchet derivative with respect to a cone exists, then the derivative with respect to the cone also exists and these derivatives coincide.

We shall call the operator A *continuously differentiable* on the set $G \subset E$ if its corresponding derivative $A'(x)$ in the norm of operators continuously depends on x:

$$\lim_{\|x - x^*\| \to 0} \|A'(x) - A'(x^*)\| = \lim_{\|x - x^*\| \to 0} \left\{ \sup_{\|h\| \leqslant 1} \|A'(x)h - A'(x^*)h\| \right\} = 0\,.$$

The concept of uniform continuous differentiability is defined in a natural way.

3.1.3. *Derivatives with respect to non-oblate cones*

A natural question arises concerning those assumptions for which the existence of the Fréchet derivative follows from the existence of deriva-

tives with respect to a cone. The statement proved below is analogous to the well-known theorem of the differential calculus that the continuity of the partial derivatives implies the existence of the total differential for functions of many variables.

We shall say that the cone K is *non-oblate* if a number n can be found such that an element $u=u(x)\in E$ can be found for every $x\in E$ such that

$$x\leqslant u(x) \tag{3.11}$$

and

$$\|u(x)\|\leqslant n\|x\| \qquad (x\in E)\,. \tag{3.12}$$

It is not difficult to see that the cones of non-negative functions are non-oblate in the spaces C and L_p. It turns out that *the class of non-oblate cones coincides with the class of reproducing cones.** In the proof only the fact that every reproducing cone is non-oblate is necessary.

If K is a reproducing cone, then E is the union of the sets E_n consisting of the elements x for which an element $u(x)$ can be found such that the relations (3.11) and (3.12) are satisfied. Since E is of the second category, then some set E is everywhere dense in some spherical shell $T\{r\leqslant\|x-x_0\|\leqslant R\}$. Let $-x_0\leqslant y_0\in K$ and $\|y_0\|\leqslant n_2\|x_0\|$. Then for every element $x-x_0 (x\in E_{n_1}\cap T)$ an element $u(x)+y_0\in K$ can be found such that $x-x_0\leqslant u(x)+y_0$ and

$$\|u(x)+y_0\|\leqslant n_1\|x\|+n_2\|x_0\|\leqslant(n_1+n_2)\|x_0\|+n_1\|x-x_0\|\leqslant n_3\|x-x_0\|\,.$$

This means that the set E_{n_3} is everywhere dense in the spherical shell $r\leqslant\|x\|\leqslant R$. But each of the sets E_n contains together with every point x

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* It follows from the non-oblateness of the reproducing cone that the positive additive functional f is continuous on the reproducing cone. Assuming the contrary (and by virtue of the non-oblateness of the cone!) we can find a sequence $x_n\in K$ such that

$$0<\|x_n\|<\frac{1}{n^3} \quad \text{and} \quad f(x_n)\to\infty\,.$$

We introduce for consideration the element

$$z=x_1+2x_2+3x_3+\ldots+nx_n+\ldots$$

It is obvious that $\theta\leqslant nx_n\leqslant z$ and $f(nx_n)\leqslant f(z)$, from which it follows that

$$f(x_n)\leqslant\frac{1}{n}f(z) \qquad (n=1,2,\ldots)\,.$$

We have arrived at a contradiction.

This argument reduces to a generalization of Theorem 2.3 indicated on p. 64: if the cone K_1 is reproducing and the cone K_2 is normal, then it suffices for the continuity of an additive operator A that the condition $AK_1\subset K_2$ be satisfied.

all of the ray $tx(t \geqslant 0)$. Therefore E_{n_3} is dense in all of E. We shall show that $E_{2n_3} = E$. We construct with respect to the element $x \in E$ the elements x_1 and y_1 such that

$$x_1 \in E_{n_3}, \quad y_1 \in K, \quad \|x_1\| = \|x\|, \quad \|x - x_1\| \leqslant \tfrac{1}{2}\|x\|, \quad x_1 \leqslant y_1,$$
$$\|y_1\| \leqslant n_3 \|x_1\|.$$

Analogously, we construct with respect to the element $x - x_1$ an x_2 and a y_2 such that

$$x_2 \in E_{n_3}, \quad y_2 \in K, \quad \|x_2\| = \|x - x_1\| \leqslant \tfrac{1}{2}\|x\|,$$
$$\|x - x_1 - x_2\| \leqslant \tfrac{1}{4}\|x\|, \quad x_2 \leqslant y_2, \quad \|y_2\| \leqslant n_3 \|x_2\|.$$

Moreover, we construct the sequence of x_k and y_k such that

$$x_k \in E_{n_3}, \quad y_k \in K,$$
$$\|x_k\| = \|x - x_1 - \ldots - x_{k-1}\| \leqslant 2^{-k+1}\|x\|,$$
$$\|x - x_1 - \ldots - x_k\| \leqslant 2^{-k}\|x\|, \quad x_k \leqslant y_k,$$
$$\|y_k\| \leqslant n_3 \|x_k\| \qquad (k = 1, 2, \ldots).$$

Obviously, the sequences

$$z_k = x_1 + x_2 + \ldots + x_k, \quad u_k = y_1 + y_2 + \ldots + y_k \qquad (k = 1, 2, \ldots)$$

converge: the first converges to the element x, the second converges to some element $u(x) \in K$. In this connection,

$$x \leqslant u(x), \qquad \|u(x)\| \leqslant \lim_{k \to \infty} \sum_{i=1}^{k} \|y_i\| \leqslant 2n_3 \|x\|.$$

Thus, $E_{2n_3} = E$. Our statement is thus proved.*

Locally compact cones in infinite-dimensional spaces (see p. 33) can serve as an example of oblate cones. We prove this by contradiction.

Let the cone K be locally compact and non-oblate. We then consider an arbitrary bounded in norm sequence $x_n \in E (n = 1, 2, \ldots)$. According to our assumption, a sequence of elements $u(x_n) \in K$ can be found which satisfy the conditions

$$x_n \leqslant u(x_n), \quad \|u(x_n)\| \leqslant M\|x_n\| \qquad (n = 1, 2, \ldots).$$

It follows from the local compactness of the cone K that a converging subsequence $u(x_n)$ can be chosen from the sequence $u(x_{n_i})$. The elements $u(x_{n_i}) - x_{n_i}$ $(i = 1, 2, \ldots)$ also belong to K and are bounded in norm; there-

* With analogous arguments, V. Ya. Stecenko has shown that every K-reproducing cone K_0 (see chapter 1, section 1.8) is non-oblate in the sense that every element $x \in E$ is representable in the form $x = u - v (u \in K_0, v \in K)$, where $\|u\| \leqslant M\|x\|$ (see I. A. Bakhtin, M. A. Krasnosel'skii, V. Ya. Stecenko [1]).

fore, a new subsequence of indices n_j can be chosen such that the sequence $u(x_{n_j})-x_{n_j}$ converges. This means that the sequence x_{n_j} is convergent. We have shown that from every bounded in norm sequence a convergent subsequence can be chosen. This contradicts the infinite dimensionality of E.

Theorem 3.1. *Let a weak derivative $A'(x)$ in the non-oblate cone K be continuous in the norm of the operators in a neighborhood of the point x_0. Then $A'(x)$ is a strong Fréchet derivative of the operator A at the points $x\in U$.*

Proof. We introduce the notation

$$\omega(x, h)=A(x+h)-Ax-A'(x)h\,,$$

where x, $x+h$, $x+h+u(-h)\in U$. It is obvious that

$$\omega(x, h)=A(x+h)-A[x+h+u(-h)]+A[x+h+u(-h)]-Ax-A'(x)h\,,$$

and by virtue of (3.8), since $u(-h)\in K$ and $h+u(-h)\in K$

$$\begin{aligned}\omega(x, h) = &-\int_0^1 A'[x+h+tu(-h)]u(-h)\,dt+\\ &+\int_0^1 A'[x+th+tu(-h)][h+u(-h)]dt-\int_0^2 A'(x)h\,dt=\\ =&\int_0^1\{A'(x)-A\,[x+h+tu(-h)]\}u(-h)+\\ &+\int_0^1\{A'[x+th+tu(-h)]-A'(x)\}u(-h)+\\ &+\int_0^1\{A'[x+th+tu(-h)]-A'(x)\}h\,dt\,.\end{aligned}$$

The norms of the differences in the square brackets in all of the last three integrals tend to zero as $\|h\|\to 0$ (by virtue of (3.12)). Therefore (again we make use of (3.12))

$$\|\omega(x, h)\|-o(\|h\|)\,.$$

This completes the proof of the theorem.

3.1.4. *Derivatives with respect to a cone of completely continuous operators* If a completely continuous operator A is strongly differentiable according to Fréchet, then its derivative $A'(x)$ is a linear completely continuous operator (see M. A. Krasnosel'skii [5]). This statement carries over to operators which are differentiable with respect to a cone.

Lemma 3.1. *Let a completely continuous operator A have a strong Fréchet derivative $A'(x_0)$ at the point x_0 with respect to the cone K.*
Then the derivative $A'(x)$ transforms every bounded set $T \subset K$ into a compact set.

Proof. Since the operator $A'(x_0)$ is linear, it suffices to consider the case where T is the intersection of the unit sphere with K.
We assume that the statement of the Lemma is false. Then a sequence of elements $h_n \in K$ $(n=1, 2, \ldots)$ can be found such that

$$\|h_n\| = 1 \; ; \quad \|A'(x_0)h_i - A'(x_0)h_i\| \geqslant \varepsilon_0 , \qquad (i \neq j) ,$$

where ε_0 is some positive number.
We choose a $\delta > 0$ such that

$$\|A(x_0+h) - Ax_0 - A'(x_0)h\| \leqslant \frac{\varepsilon}{3} \|h\|$$

for $h \in K$ and $\|h\| \leqslant \delta$. Then for $i \neq j$,

$$\begin{aligned} \|A(x_0+\delta h_i) - A(x_0+\delta h_j)\| &\geqslant \delta \|A'(x_0)h_i - A'(x_0)h_j\| - \\ &- \|A(x_0+\delta h_i) - Ax_0 - \delta A'(x_0)h_i\| - \\ &- \|A(x_0+\delta h_j) - Ax_0 - \delta A'(x_0)h_j\| \geqslant \frac{\varepsilon_0 \delta}{3} . \end{aligned}$$

The inequality obtained means that the set AT is not compact. We have arrived at a contradiction.
This completes the proof of the Lemma.

With further assumptions about the structure of the cone, it can be shown that under the conditions of Lemma 3.1, the operator $A'(x_0)$ is completely continuous.

Theorem 3.2. *The strong Fréchet derivative in a non-oblate cone K of a completely continuous operator is a completely continuous operator.*

Proof. By virtue of (3.11) and (3.12) every element $x \in E$ is representable in the form

$$x = u - v \qquad (u, v \in K) ,$$

where

$$\|u\| = \|u(x)\| \leqslant M\|x\| , \qquad \|v\| \leqslant \|x\| + \|u\| \leqslant (1+M)\|x\| .$$

This means that every element $A'(x_0)x$ $(\|x\| \leqslant 1)$ is representable in the form

$$A'(x_0)x = A'(x_0)u - A'(x_0)v \qquad (u, v \in K , \quad \|u\| \leqslant M , \quad \|v\| \leqslant 1+M) .$$

By virtue of Lemma 3.1, the elements $A'(x_0)u$ and $A'(x_0)v$ belong to

compact sets; therefore their differences also form a compact set.
This completes the proof of the theorem.

3.1.5. *Positive and monotone operators*
The non-linear operator A is called *monotone* on the set $T \subset E$ if it follows from $x \leqslant y (x, y \in T)$ that $Ax \leqslant Ay$.
The following lemma gives a criterion for monotoneity.

Lemma 3.2. *Let the set T be convex. Let the operator A be differentiable at every point of the set T in the directions of the cone K where the values of all derivatives of the function $A[x+t(y-x)]$ $(x, y \in T, x \leqslant y)$ belong to K. Then the operator A is monotone on T.*

Proof. Assuming the contrary, we find elements $x, y \in T$ such that $x \leqslant y$, but $Ax \not\leqslant Ay$. We then construct a positive linear functional $f(x)$ on K such that

$$f(Ax) > f(Ay) .$$

It follows from this inequality that for some $t_0 \in [0, 1]$ the derivative of the function

$$\alpha(t) = f[A(x+t(y-x))] \qquad (0 \leqslant t \leqslant 1)$$

will be negative. From this in turn it follows that $y'(t_0) \notin K$, where

$$y(t) = A[x+t(y-x)] \qquad (0 \leqslant t \leqslant 1) .$$

We have arrived at a contradiction.
This completes the proof of the lemma.

Lemma 3.2 is reversible in a definite sense: if the operator A, differentiable in the directions of the cone, is monotone, then all the derivatives of the functions $A[x+t(y-x)](x, y \in T, x \leqslant y)$ are positive, that is, belong to the cone. The proof is obvious.
The operator A on the set T is positive if $AT \subset K$. It is obvious that a monotone operator A on T will be positive if $A\theta \in K$.
We observe that it follows from the positiveness on the cone K of the operator A and from $A\theta = \theta$ that $[A(th)]'_{t=0} \in K$ for $h \in K$ (if the corresponding derivatives exist).
We remark in conclusion that for the operators, differentiable with respect to a cone, membership of all derivatives of the functions $A(x+th)$ $(h \in K)$ to the cone is equivalent to the operator $A'(x)$ leaving the cone K invariant.

3.2. Derivatives at infinity

3.2.1. *Definitions*

We shall say that the function $y(t)$ with values in E is *differentiable at infinity* if the ratio $(1/t)y(t)$ converges to some element $y'(\infty)\in E$ as $t\to+\infty$. As usual, we can speak about *strong* or *weak differentiability at infinity* depending on whether $(1/t)y$ converges strongly or weakly to $y'(\infty)$.

We consider the operator A, acting in the space E and defined on elements with a sufficiently large norm. We shall say that the operator A is (strongly or weakly) *differentiable at infinity in the direction* $h\in E$ if the function

$$y(t)=A(th) \tag{3.13}$$

is differentiable at infinity. It is clear that from the differentiability in the direction h follows the differentiability in all directions kh $(k>0)$ where

$$[A(tkh)]'_{t=\infty}=k[A(th)]'_{t=\infty}\,.$$

We assume that the operator A is differentiable in all directions $h\in E$ $(h\neq\theta)$ where the derivatives $y'(\infty)$ of the functions (3.13) are representable in the form

$$y'(\infty)=A'(\infty)h \qquad (h\in E)\,, \tag{3.14}$$

where $A'(\infty)$ is some continuous linear operator. Then the operator A will be called (strongly or weakly) *differentiable at infinity*. We shall call the operator $A'(\infty)$ the (strong or weak) *derivative at infinity* of the operator A.

We assume that the operator A is differentiable at infinity in the directions $h\in K$ $(h\neq\theta)$ and the derivatives of the functions (3.13) allow the representation (3.14) for $h\in K$. In this case, we shall say that A is (strongly or weakly) *differentiable at infinity in the cone*. In this connection, we shall call $A'(\infty)$ the *derivative at infinity with respect to the cone*.

It is clear that differentiability at infinity with respect to a cone is an essentially lesser restriction than ordinary differentiability. The operator

$$A\{\xi,\eta\}=\{e^{-(\xi+\eta)},0\}\,, \tag{3.15}$$

which is non-differentiable at infinity but has a derivative at infinity equal to zero with respect to the cone K can serve as the simplest example in the two-dimensional space of points $\{\xi,\eta\}$ with the cone K of elements with non-negative coordinates.

We shall call the operator $A'(\infty)$ a *strong asymptotic derivative* of the operator A and we shall call the operator A *strongly asymptotically linear* if

$$\lim_{R\to\infty} \sup_{\|x\|\geq R} \frac{\|Ax - A'(\infty)x\|}{\|x\|} = 0 . \tag{3.16}$$

Sometimes the following simple remark is useful: *if the operator A is strongly asymptotically linear, then all the operators C of the form*

$$Cx = A(x_0 + x)$$

are strongly asymptotic linear where x_0 is a constant element and where $A'(\infty)$ is a strong asymptotic derivative of all the operators C. The proof follows from the evident inequalities:

$$\begin{aligned} &\sup_{\|x\|\geq R} \frac{\|A(x_0+x) - A'(\infty)x\|}{\|x\|} \leqslant \\ &\leqslant \sup_{\|x\|\geq R-\|x_0\|} \frac{\|A(x_0+x) - A'(\infty)(x_0+x)\| + \|A'(\infty)x_0\|}{\|x_0+x\|} \leqslant \\ &\leqslant \sup_{\|x\|\geq R-2\|x_0\|} \frac{\|Ax - A'(\infty)x\|}{\|x\|} + \|A'(\infty)\| \frac{\|x_0\|}{R-2\|x_0\|} . \end{aligned} \tag{3.17}$$

We call the operator $A'(\infty)$ a *strong asymptotic derivative with respect to the cone K* and the operator A *strongly asymptotically linear with respect to the cone K* if

$$\lim_{R\to\infty} \sup_{\|x\|\geq R,\, x\in K} \frac{\|Ax - A'(\infty)x\|}{\|x\|} = 0 . \tag{3.18}$$

The remark proved above about the operators $C_x = A(x_0 + x)$ remains valid for the derivatives in the cone if $x_0 \in K$; the inequalities (3.17) remain valid for the elements $x \in K$.

If, in the conditions (3.16) and (3.18) the relations $\|Ax - A'(\infty)x\|/\|x\|$ are replaced by $l[Ax - A'(\infty)x]/\|x\|$ where $l(x)$ is an arbitrary linear continuous functional, then we obtain, in a natural way, the definitions of *weak asymptotic linearity* and *weak asymptotic linearity in the cone.*

We remark in conclusion that all the definitions mentioned above, connected with differentiability at infinity, carry over without modification to operators A acting from one Banach space into another.

3.2.2. *Existence of a strong asymptotic derivative*

Theorem 3.3. *Let the operator A have a weak Gâteaux derivative $A'(x)$ at every point x with a sufficiently large norm. Let*

$$\lim_{\|x\|\to\infty} \|A'(x) - B\| = 0 , \tag{3.19}$$

where B is some linear operator.

Then B *is a strong asymptotic derivative of the operator* A.

Proof. By virtue of (3.19) an $R_0>0$ can be found such that

$$\|A'(x)\| \leqslant M \qquad (\|x\| \geqslant R_0)\,. \tag{3.20}$$

We first show that for an arbitrary $R \geqslant R_0$

$$\sup_{\|x\|=R} \|Ax\| = a(R) < \infty\,. \tag{3.21}$$

Let $\|x_0\| = R$. Then a number n_0 can be found such that for every x whose norm is equal to R either the interval $[2x, n_0 x_0]$ or the interval $[2x, -n_0 x_0]$ consists entirely of points whose norm is greater than R. In fact, assuming the contrary, we could find a sequence of elements x_n $(n=1, 2, \ldots)$ and sequences of numbers $\alpha_n, \beta_n \in [0, 1]$, such that

$$\left.\begin{aligned} &\|x_n\| = R\,, \\ &\|2\alpha_n x_n + (1-\alpha_n) n x_0\| \leqslant R\,, \\ &\|2\beta_n x_n - (1-\beta_n) n x_0\| \leqslant R \\ &\qquad (n=1, 2, \ldots)\,. \end{aligned}\right\} \tag{3.22}$$

Then

$$\|(1-\beta_n)[2\alpha_n x_n + (1-\alpha_n) n x_0] + \\ + (1-\alpha_n)[2\beta_n x_n - (1-\beta_n) n x_0]\| \leqslant (2-\alpha_n-\beta_n) R\,,$$

that is

$$2\alpha_n + 2\beta_n - 4\alpha_n\beta_n \leqslant 2-\alpha_n-\beta_n \qquad (n=1, 2, \ldots)\,.$$

The last inequality cannot be satisfied* since by virtue of (3.22) $\alpha_n \to 1$ and $\beta_n \to 1$, where $\alpha_n < 1$ and $\beta_n < 1$.

Let, for definiteness, the interval $[2x, n_0 x_0]$ consist of points whose norm is greater than R. Then, at all the points of this interval, the operator A has a derivative satisfying the inequalities (3.20). Consequently,

$$\|A(2x) - A(n_0 x_0)\| \leqslant M\|2x - n_0 x_0\|\,.$$

Analogously,

$$\|Ax - A(2x)\| \leqslant M\|x\|\,.$$

Thus,

$$\|Ax\| \leqslant \|Ax - A(2x)\| + \|A(2x) - A(a_0 x_0)\| + \|A(n_0 x_0)\| \leqslant \\ \leqslant M(3+n_0) R + \|A(n_0 x_0)\|\,.$$

---

* We consider the function $\varphi(\alpha, \beta) = 3\alpha + 3\beta - 4\alpha\beta - 2$. It is obvious that $\varphi(1, 1) = 0$ and $\varphi_\alpha'(1, 1) = \varphi_\beta'(1, 1) = -1$. Therefore, for the values of α, β near to 1 but less than 1, the inequality $\varphi(\alpha, \beta) > 0$ is satisfied, that is

$$2\alpha + 2\beta - 4\alpha\beta > 2 - \alpha - \beta\,.$$

If the interval $[2x, -n_0 x_0]$ consists of points whose norm is greater than R then the inequality

$$\|Ax\| \leqslant M(3+n_0)R + \|A(-n_0 x_0)\|$$

is satisfied.
This completes the proof of the inequality (3.21).

In order to prove the equality (3.16), it suffices to show that for every linear functional $f(x)$ the equality

$$\lim_{R\to\infty} \sup_{\|x\|\geqslant R} \frac{f(Ax-Bx)}{\|f\|\,\|x\|} = 0$$

is valid.
Let $\varepsilon>0$ be given. We choose an $R_1 \geqslant R_0$ such that

$$\|A'(x)-B\| \leqslant \frac{\varepsilon}{2} \qquad (\|x\| \geqslant R_1)\,.$$

We consider the scalar function

$$\psi(t)=f[A(tx)-tBx] \qquad \left(\frac{R_1}{\|x\|} \leqslant t \leqslant 1\right).$$

Obviously,

$$\psi'(t)=f\{[A'(tx)-B]x\}\,,$$

from which it follows that

$$|\psi'(t)| \leqslant \frac{\varepsilon}{2}\|f\|\,\|x\|$$

and, by virtue of the Lagrange theorem,

$$\left|f\left[Ax-A\left(\frac{R_1}{\|x\|}x\right)-B\left(x-\frac{R_1}{\|x\|}x\right)\right]\right| =$$

$$= \left|\psi(1)-\psi\left(\frac{R_1}{\|x\|}\right)\right| \leqslant \frac{\varepsilon}{2}\|f\|\,\|x\|\,.$$

We then choose a number $R_2 \geqslant R_1$ in order for the inequality

$$\frac{a(R_1)+R_1\|B\|}{R_2} < \frac{\varepsilon}{2}$$

to be satisfied. Then for $\|x\| \geqslant R_2$,

$$\frac{|f(Ax-Bx)|}{\|f\|\,\|x\|} \leqslant \frac{\left|f\left[Ax-A\left(\frac{R_1}{\|x\|}x\right)-B\left(x-\frac{R_1}{\|x\|}x\right)\right]\right|}{\|f\|\,\|x\|} +$$

$$+\frac{\left|f\left[A\left(\frac{R_1}{\|x\|}x\right)\right]\right| + \frac{R_1}{\|x\|}\,|f(Bx)|}{\|f\|\,\|x\|} \leqslant \frac{\varepsilon}{2} + \frac{a(R_1)+R_1\|B\|}{\|x\|} < \varepsilon\,.$$

This completes the proof of the theorem.

The proof would be simplified if in addition it were required that the operator A be defined on all E and that for every R (3.21) is satisfied.

3.2.3. *Strongly asymptotic linear operators with respect to a cone*

Theorem 3.4. *Let the operator A be defined on the cone K and satisfy the condition*

$$\sup_{x\in K,\ \|x\|\leqslant R} \|Ax\| < \infty \tag{3.23}$$

for every R. Let A have at every point $x\in K$ with a sufficiently large norm a weak Gâteaux derivative $A'(x)$ with respect to the cone where

$$\lim_{x\in K,\ \|x\|\to\infty} \|A'(x)-B\| = 0\,, \tag{3.24}$$

where B is some linear operator.

Then B is a strong asymptotic derivative with respect to the cone K of the operator A.

For the proof it suffices to repeat the last part of the proof of Theorem 3.3 observing first that the condition (3.23) replaces (3.21).

We now assume that the operator A is completely continuous. Its strong asymptotic derivative (if it exists) will be a completely continuous linear operator (see M. A. Krasnosel'skii [5]). This statement carries over to the derivatives with respect to a cone: *the strong asymptotic derivative in the cone K of a completely continuous operator is a linear operator which transforms every bounded set $T\in K$ into a compact set.* This statement is proved as Lemma 3.1 is proved.

It follows from the last statement (analogous to how Theorem 3.2 followed from Lemma 3.1) that *the strong asymptotic derivative in a non-oblate cone of a completely continuous operator is a completely continuous operator.*

It was noted above (Sections 3.1 and 3.5) that the derivatives with respect

to a cone of monotonic operators are positive operators. An analogous statement is

Lemma 3.3. *The derivative $A'(\infty)$ with respect to a cone of the positive operator A is a linear positive operator.*

Proof. We assume that $A'(\infty)$ is a weak derivative with respect to a cone. We assume that $A'(\infty)h_0 \in K$, where h_0 is some element of K.
We construct a linear positive functional $f(x)$ such that

$$\alpha_0 = f[A'(\infty)h_0] < 0 .$$

Then the function

$$\chi(t) = f[A(th_0)]$$

will assume negative values for large values of t since

$$\lim_{t\to\infty} \chi(t) = \lim t \left\{ f\left[\frac{A(th_0) - tA'(\infty)h_0}{t}\right] + \alpha_0 \right\} = -\infty . \tag{3.25}$$

This means that $A(th_0) \in K$ for large t, that is, the operator A does not have the property of positiveness.
This completes the proof of the lemma.

3.3. Inequalities for elements with small norms

3.3.1. *Formulation of the problem*
Let A be some non-linear operator and let $A\theta = \theta$.
In the sequel, the elements $x_0 \in K$ such that $Ax_0 \leqslant x_0$ or $Ax_0 \geqslant x_0$ play an essential role. If the elements x_0 are sought satisfying the stated inequalities with small norms, then naturally the derivative $A'(\theta)$ of the operator A at the point θ is used. It can be expected, for instance, that the elements x_0 with small norms for which $A'(\theta)x_0 = \lambda_0 x_0$, where $\lambda_0 > 1$, possess the property that $Ax_0 \geqslant x_0$. However, if $A'(\theta)x_0 = \lambda_0 x_0$ and $\lambda_0 < 1$, then it can be expected that the inequality $Ax \leqslant x_0$ will be satisfied. It turns out that in several cases the stated propositions are valid; these cases are described in the following subsections.
It is important to know that in other problems the relations $x \lessgtr Ax$ or $x \gtrless Ax$ are satisfied for all non-zero elements $x \in K$ with sufficiently small norms. With respect to the linear operators, the question about such relations is considered in chapter 2 (Section 2.5, subsection 2.5.4). It is

natural, in the case of the non-linear operators, to reduce the question about the relations which are of interest to us to the consideration of the operator $A'(\theta)$. Cases will be stated below when this can be done.

3.3.2. *"Progressing" elements*

We shall say that the element x_0 "progresses" upon application of the operator A if

$$Ax_0 \geqslant x_0 . \tag{3.26}$$

We shall denote by $K(r)$ the collection of elements $x \in K$ such that $\|x\| \leqslant r$.

Lemma 3.4. *Let K be a solid cone. Let the operator $A(A\theta = \theta)$ defined on $K(r)$ have a derivative $A'(\theta)$ with respect to the cone at the point θ. Let the linear operator $A'(\theta)$ have a characteristic vector h_0 which is an interior element of the cone K where*

$$A'(\theta)h_0 = \lambda_0 h_0 , \tag{3.27}$$

where $\lambda_0 > 1$.

Then an $\varepsilon_0 > 0$ can be found such that the inequality (3.26) *is satisfied for all the elements $x_0 = \varepsilon h_0$ for $0 < \varepsilon \leqslant \varepsilon_0$.*

Proof. Assuming the contrary, we can find a sequence $\varepsilon_n \to 0$ such that

$$A(\varepsilon_n h_0) \gtrless \varepsilon_n h_0 ,$$

that is

$$\frac{A(\varepsilon_n h_0)}{\varepsilon_n} - \lambda_0 h_0 + (\lambda_0 - 1)h_0 \overline{\in} K \qquad (n = 1, 2, \ldots) . \tag{3.28}$$

It follows from the differentiability of the operator A in the direction h_0 and from (3.27) that

$$\lim_{n \to \infty} \left\| \frac{A(\varepsilon_n h_0)}{\varepsilon_n} - \lambda_0 h_0 \right\| = \lim_{\varepsilon_n \to 0} \frac{\|A(\varepsilon_n h_0) - \varepsilon_n A'(\theta)h_0\|}{\varepsilon_n} = 0 .$$

But then, by virtue of (3.28), the element $(\lambda_0 - 1)h_0$ is the limit of a sequence of elements lying outside of the cone. This means that the element $(\lambda_0 - 1)h_0$ and, together with it the element h_0, is not an interior element of K. We have arrived at a contradiction.

This completes the proof of the lemma.

If the cone K does not have the property of solidity, then the statement of Lemma 3.4 is not true in the general case (even if the operator A is

positive). We consider, for example, in the space L_p (where p is some fixed number of $[1, \infty)$) of the functions $x(t)$ on the closed interval $[0, 1]$ the operator

$$Ax(t) = 2t^{\|x\|^2} \int_0^1 x(s)\,ds\,.$$

The positiveness of this operator is evident. The operator A has the derivative

$$A'(\theta)x(t) = 2\int_0^1 x(s)\,ds\,,$$

at the point θ since $A\theta = \theta$ and for $x(t) \in K$

$$\|Ax - A'(\theta)x\| = 2\int_0^1 x(s)\,ds \left\{\int_0^1 (1 - t^{\|x\|^2})^p\,dt\right\}^{1/p} \leqslant$$

$$\leqslant 2\int_0^1 x(s)\,ds \left\{\int_0^1 (1 - t^{\|x\|^2})\,dt\right\}^{1/p} =$$

$$= 2\int_0^1 x(s)\,ds \left(\frac{\|x\|^2}{1 + \|x\|^2}\right)^{1/p} = o(\|x\|)\,.$$

The operator $A'(\theta)$ has a characteristic function $h_0(t) \equiv 1$, to which the characteristic value $\lambda_0 = 2$ corresponds. At the same time, for an arbitrarily small positive ε the inequality $A(\varepsilon h_0) \geqslant \varepsilon h_0$ is not satisfied because $A(\varepsilon h_0) = 2\varepsilon t^{\varepsilon^2}$ and for small numbers t this function assumes values less than ε. This example and Theorem 3.4 show that in the determination of the element x_0 satisfying the condition (3.26), it is convenient to make use of norms (if this is possible) in which the cone K is solid. In particular, it is appropriate to use different u_0-norms.

We mention one of the statements which are obtained in this connection where we include with it for the convenience of future reference the Lemma 3.4:

Theorem 3.5. *Let the operator $A(A\theta = \theta)$ defined on some $K(r)$ have a strong derivative $A'(\theta)$ with respect to a cone. Let*

$$A'(\theta)h_0 = \lambda_0 h_0\,, \tag{3.27}$$

where $h_0 \in K (h_0 \neq \theta)$ and $\lambda_0 > 1$. Finally, let either one of the following two conditions be satisfied:

3.5(a) *The cone K is solid; h_0 is an interior element of K;*

3.5(b) *The operator A is differentiable in the direction h_0 in the h_0-norm:*

$$\lim_{\varepsilon \to 0} \frac{\|A(\varepsilon h_0) - \varepsilon A'(\theta)h_0\|_{h_0}}{\varepsilon} = 0\,. \tag{3.29}$$

Then an $\varepsilon_0 > 0$ *can be found such that all the elements* $x_0 = \varepsilon h_0$ *satisfy the inequality*

$$Ax_0 \geqslant x_0 , \tag{3.26}$$

if $0 < \varepsilon \leqslant \varepsilon_0$.

Proof. The case 3.5(a) corresponds to the Lemma 3.4. For condition 3.5(b) the proof is carried out analogously with the following evident modifications: from (3.28), it is necessary to conclude that $(\lambda_0 - 1)h_0$ is not an interior element of the cone $\overline{K}_{h_0}$ in the space $\bar{E}_{h_0}$; this contradicts Lemma 1.4 and Theorem 1.4.
This completes the proof of the theorem.

The differentiability of the operator A in the conditions of Theorem 3.5 can be replaced by a less restricted assumption in which the operator A is differentiable only in the direction h_0 and the derivative in this direction is equal to $\lambda_0 h_0$ where $\lambda_0 > 1$. Such an assumption is not convenient since it does not give a formula either for the determination of the element h_0 or for the proof of the existence of such an element.

3.3.3. *Use of higher derivatives*
If in the condition (3.27) $\lambda_0 < 1$, then, as will be shown below, even a positive operator A, generally speaking, does not have in a cone elements with a small norm which "progress" upon application of the operator A. The case $\lambda_0 = 1$ remains doubtful. In this case, one cannot determine the presence of elements which "progress" by the derivative $A'(\theta)$.
In this subsection, we shall assume that the operator A allows the representation

$$Ax = A'(\theta)x + Cx + Dx , \tag{3.30}$$

where $A'(\theta)$ is a strong derivative with respect to the cone K and C is a homogeneous operator of degree $k (k > 1)$:

$$C(\varepsilon x) = \varepsilon^k Cx \qquad (x \in K) , \tag{3.31}$$

and D is an operator of a higher order of smallness

$$\lim_{\varepsilon \to 0} \frac{\|D(\varepsilon x)\|}{\varepsilon^k} = 0 \qquad (x \in K) . \tag{3.32}$$

Theorem 3.6. *Let the operator* A *allow the representation* (3.30) *where*

$$A'(\theta)h_0 = h_0 \tag{3.33}$$

and $Ch_0 \in K$. *Let either one of the following two conditions be satisfied*:
3.6(a) *The cone* K *is solid*; Ch_0 *is an interior element of* K;
3.6(b) *The operator* A *satisfies the conditions*

$$\lim_{\varepsilon \to 0} \frac{\|D(\varepsilon h_0)\|_{h_0}}{\varepsilon^k} = 0\,, \tag{3.34}$$

$$\alpha h_0 \leqslant Ch_0 \leqslant \beta h_0\,, \tag{3.35}$$

where $\alpha, \beta > 0$.
Then an $\varepsilon_0 > 0$ *can be found such that all the elements* $x_0 = \varepsilon h_0$ *satisfy the inequality* (3.26) *if* $0 < \varepsilon \leqslant \varepsilon_0$.

Proof. We consider the case 3.6(b) which includes the case 3.6(a).
Assuming the contrary, we can find a sequence $\varepsilon_n \to 0$ such that

$$A(\varepsilon_n h_0) \overline{\geqslant} \varepsilon_n h_0 \qquad (n = 1, 2, \ldots)\,,$$

that is

$$Ch_0 + \frac{D(\varepsilon_n h_0)}{\varepsilon_n^k} \overline{\in} K \qquad (n = 1, 2, \ldots)\,.$$

It follows from here by virtue of (3.34) that Ch_0 is not an interior element of the cone $\overline{K}_{h_0}$ in the space $\overline{E}_{h_0}$. On the other hand, the condition (3.35) means that Ch_0 is an interior element of the cone $\overline{K}_{h_0}$. We have arrived at a contradiction.
This completes the proof of the theorem.

3.3.4. *Use of minorants*
We show still one more simple method for the determination of elements x_0 which "progress" upon application of the operator A.
We call the operator A^- a *minorant of the operator* A on the set T if

$$A^- x \leqslant Ax \qquad (x \in T)\,. \tag{3.36}$$

We assume that for the minorant A^- of the operator A an element x_0 was found such that $A^- x_0 \geqslant x_0$. Then, obviously,

$$Ax_0 \geqslant x_0\,.$$

Thus, we can make use of different minorants for the determination of the "progressing" element x_0. We can consider in a number of cases the derivative $A'(\theta)$ or the operator $(1-\delta)A'(\theta)$ (δ is a sufficiently small number) as a minorant even if all the conditions of Theorem 3.5 or Theorem 3.6 are not satisfied.

3.3.5. *"Retrogressing" elements*

We shall say that the element x_0 "retrogresses" upon application of the operator A if

$$Ax_0 \leqslant x_0 . \tag{3.37}$$

It is not difficult to prove the criteria, analogous to Theorems 3.5 and 3.6, for the existence in K of small non-zero "retrogressing" elements. We shall formulate these criteria in the form of a single theorem.

Theorem 3.7. *Let the operator $A(A\theta=\theta)$ defined on some $K(r)$ have a strong derivative $A'(\theta)$ with respect to the cone K where $Ah_0=\lambda_0 h_0$ and where $h_0 \in K$, $h_0 \neq \theta$.*

Then for the existence of an $\varepsilon_0>0$ such that all the elements $x_0=\varepsilon h_0$ $(0<\varepsilon\leqslant\varepsilon_0)$ "retrogress" upon application of the operator A, it suffices that any one of the following three conditions be satisfied:

3.7(a) *If $0\leqslant\lambda_0<1$, the cone K is solid; h_0 is an interior element of K;*

3.7(b) *If $0\leqslant\lambda_0<1$, the operator A is differentiable in the direction h_0 in the h_0-norm (condition* 3.29 *is satisfied);*

3.7(c) *If $\lambda_0=1$, the operator A allows the representation* (3.30) *where $\boldsymbol{D}$ satisfies the conditions* (3.34) *and*

$$\alpha h_0 \leqslant -\boldsymbol{C}h_0 \leqslant \beta h_0 , \tag{3.38}$$

where $\alpha, \beta>0$.

We do not go into the proof.

We make one further remark.

We call the operator A^+ a *majorant of the operator A* on the set T if

$$A^+x \geqslant Ax \qquad (x\in T) . \tag{3.39}$$

It is not difficult to see that "retrogressing" elements x_0 upon the application of the majorant A^+ and upon application of the operator A will satisfy the condition (3.36).

3.3.6. *Absence of "progressing" elements*

We shall say that the operator A is *uniformly h_0-bounded above* if for every sufficiently small number $\varepsilon>0$ an $r=r(\varepsilon)>0$ can be found such that

$$Ax \leqslant \varepsilon h_0 \qquad (x\in K, \|x\|\leqslant r) . \tag{3.40}$$

If the cone K is solid and h_0 is an interior element of the cone, then all the continuous operators A at the point θ satisfying the condition $A\theta=\theta$ will be uniformly h_0-bounded above.

We call the point θ *attracting* for the operator A if $A\theta=\theta$ and if an $r_0>0$

can be found such that

$$Ax \gtrdot x \qquad (x \in K(r_0),\ x \neq \theta)\,. \tag{3.41}$$

We mention some criteria which enable us to judge whether or not the point θ is attracting.

We denote by $\langle x_0, y_0 \rangle$ the set of $x \in E$ such that

$$x_0 \leqslant x \leqslant y_0\,.$$

Theorem 3.8. *Let $A'(\theta)$ be a strong Fréchet derivative with respect to a cone of the operator A which is uniformly h_0-bounded above, monotone and satisfies the conditions* 3.7(a) *or* 3.7(b) *of Theorem* 3.7 *where for the condition* 3.7(b) *it is assumed that λ_0 is the greatest characteristic value of the operator $A'(\theta)$.*

Then in order for the point θ to be attracting, it suffices that any one of the following conditions be satisfied:

3.8(a) *The operator A is completely continuous, the cone K is normal;*

3.8(b) *The space E is weakly complete, the unit sphere in the space E is weakly compact, the operator A is weakly continuous, the cone K allows plastering.*

Proof. We assume that the statement of the theorem is not true. Then a sequence $x_n \in K$ can be found such that

$$Ax_n \geqslant x_n \qquad (n = 1, 2, \ldots) \tag{3.42}$$

and

$$\lim_{n \to \infty} \|x_n\| = 0\,.$$

It follows from the last equality and from the uniform h_0-boundedness of the operator A that

$$Ax_n \leqslant \varepsilon_n h_0 \qquad (n = 1, 2, \ldots)\,, \tag{3.43}$$

where $\varepsilon_n \to 0$. By virtue of Theorem 3.7, we can assume that

$$A(\varepsilon_n h_0) \leqslant \varepsilon_n h_0 \qquad (n = 1, 2, \ldots)\,. \tag{3.44}$$

We consider the set $\langle x_n, \varepsilon_n h_0 \rangle$. It is bounded by virtue of the normality of the cone K. It follows from the monotoneity of the operator A and the inequalities (3.42) and (3.44) that the operator A transforms $\langle x_n, \varepsilon_n h_0 \rangle$ into itself: if

$$x_n \leqslant x \leqslant \varepsilon_n h_0\,,$$

then

$$x_n \leqslant Ax_n \leqslant Ax \leqslant A(\varepsilon_n h_0) \leqslant \varepsilon_n h_0 .$$

If the condition 3.8(a) is satisfied, then, by virtue of Schauder's principle and if the condition 3.8(b) is satisfied then, by virtue of the Tikhonov principle, the operator A has on every set $\langle x_n, \varepsilon_n h_0 \rangle$ at least one fixed point y_n:

$$Ay_n = y_n \qquad (n = 1, 2, \ldots) .$$

In this connection, $\|y_n\|_{h_0} \leqslant \varepsilon_n \to 0$ and since the cone K is normal then

$$\lim_{n\to\infty} \|y_n\| = 0 . \tag{3.45}$$

We assume that the condition 3.8(a) is satisfied. By virtue of Lemma 3.1, the operator $A'(\theta)$ transforms the sequence

$$\frac{y_n}{\|y_n\|} \qquad (n = 1, 2, \ldots) \tag{3.46}$$

into a compact sequence. Without loss of generality, it can be assumed that the sequence

$$A'(\theta)\frac{y_n}{\|y_n\|} \qquad (n = 1, 2, \ldots) \tag{3.47}$$

converges strongly to some element y^* (in the contrary case we would take a subsequence). Then the sequence (3.46) also converges strongly to y^* because by virtue of the strong differentiability with respect to a cone of the operator A at the point θ we have

$$\lim_{n\to\infty} \left\| \frac{y_n}{\|y_n\|} - y^* \right\| \leqslant \lim_{n\to\infty} \frac{\|Ay_n - A'(\theta)y_n\|}{\|y_n\|} + \lim_{n\to\infty} \left\| A'(\theta)\frac{y_n}{\|y_n\|} - y^* \right\| = 0 .$$

Thus,

$$A'(\theta)y^* = y^* . \tag{3.48}$$

Let the condition 3.8(b) be satisfied. Then, without loss of generality, we can assume that the sequence (3.46) converges weakly to some element y^* which is different from θ by virtue of Lemma 1.7. The sequence (3.47) will weakly converge to an element $A'(\theta)y^*$, since every linear operator is weakly continuous. Moreover, it follows from the equality

$$\lim_{n\to\infty} \left\| A'(\theta)\frac{y_n}{\|y_n\|} - \frac{y_n}{\|y_n\|} \right\| = \lim_{n\to\infty} \frac{\|A'(\theta)y_n - Ay_n\|}{\|y_n\|} = 0$$

that y^* satisfies the equality (3.48).

Thus, it follows from the assumption that the statement of the theorem is not true that the operator $A'(\theta)$ has a positive characteristic value equal to 1.

We shall develop further arguments separately for each of the conditions of Theorem 3.7.

Let the condition 3.7(a) be satisfied. Then the cone K is solid and therefore the operator $A'(\theta)$ is a linear h_0-bounded above operator. It then follows from Theorem 2.14 that all the characteristic values of $A'(\theta)$ do not exceed λ_0 and this contradicts (3.48). The contradiction thus obtained proves Theorem 3.8.

Let the condition 3.7(b) be satisfied; then λ_0 is the greatest characteristic value of the operator $A'(\theta)$ and we have arrived at a contradiction.

This completes the proof of the theorem.

The conditions 3.7(a) and 3.7(b) were used in the proof of the theorem in order to make it possible to take advantage of the inequalities (3.44). Therefore, *the conditions* 3.7(a) *and* 3.7(b) *can be replaced by the assumption that* $A(\varepsilon h_0) \leqslant \varepsilon h_0$ *for sufficiently small numbers* $\varepsilon > 0$ *and the assumption that* 1 *is not a characteristic value of the operator* $A'(\theta)$, *respectively.*

To determine whether or not the point θ is attracting will be more complicated in the case of non-monotone operators A. Here it is convenient to make use of majorants: if the point is attracting for the majorant A^+ then it is a fortiori attracting for A. It is convenient to make use of majorants in the study of some monotone operators.

One simple criterion in which θ is an attracting point can be given in the case of cones which allow plastering.

3.3.7. *Operators, the derivatives of which are K-decomposable*

We shall say that the linear operator is *K-decomposable* if this operator B has a characteristic vector h_0 in the cone K and has an invariant subspace $E_1 \subset E$ intersecting with K only at the point θ where the subspace E_1 has defect equal to 1. This means that every element $x \in E$ is uniquely representable in the form

$$x = f(x)h_0 + x_1 \qquad (x_1 \in E_1)\,, \tag{3.49}$$

where $f(x)$ is a linear functional which satisfies the conditions $f(h_0) = 1$, $f(x_1) = 0$ for all $x_1 \in E_1$. Then

$$Bx = \lambda_0 f(x)h_0 + Bx_1\,. \tag{3.50}$$

By virtue of Theorem 2.12, every u_0-positive linear completely continuous operator can serve as an example of a K-decomposable operator.

Theorem 3.9. *Let the operator $A(A\theta=\theta)$ have a strong Fréchet derivative with respect to a cone at the point θ equal to the K-decomposable operator* (3.50). *Let the functional $f(x)$ appearing in the representation* (3.49) *be uniformly positive*:

$$f(x)\geqslant a\|x\| \qquad (x\in K)\,.$$

Let $\lambda_0<1$.
Then the point θ will be attracting for the operator A.

Proof. Assuming the contrary, we can find a sequence $x_n\in K$ such that

$$Ax_n\geqslant x_n \qquad (n=1, 2, \ldots)$$

and

$$\lim_{n\to\infty}\|x_n\|=0\,.$$

We rewrite the last inequalities in the form

$$\frac{Ax_n-Bx_n}{\|x_n\|}\geqslant\frac{x_n-Bx_n}{\|x_n\|} \qquad (n=1, 2, \ldots)$$

and compute the values of the functional $f(x)$ of the left and right members:

$$f\left(\frac{Ax_n-Bx_n}{\|x_n\|}\right)\geqslant\frac{(1-\lambda_0)f(x_n)}{\|x_n\|} \qquad (n=1, 2, \ldots)\,.$$

The left member of the last inequality tends to zero as $n\to\infty$ and the right member is bounded below by a positive number. We have arrived at a contradiction.
This completes the proof of the theorem.

3.3.8. *Absence of "retrogressing" elements*
We call the point θ *repelling* for the operator A if $A\theta=\theta$ and if a number $r_0>0$ can be found such that

$$Ax\lesseqgtr x \qquad (x\in K(r_0),\ x\neq\theta)\,. \tag{3.51}$$

We shall formulate statements, analogous to Theorems 3.7 and 3.8, about the conditions for which θ is repelling for the operator A.
The operator $A(A\theta=\theta)$ is called *h_0-bounded below* if for every non-zero $x\in K(r)$ a $\delta>0$ can be found such that $Ax\geqslant\delta h_0$; here $\delta=\delta(x)$.

Theorem 3.10. *Let $A'(\theta)$ be a strong Fréchet derivative with respect to the cone K of the operator A. Let A be monotone, h_0-bounded below and satisfy the conditions of Theorem* 3.7, *where it is assumed that* 1 *is not a positive characteristic value of the operator $A'(\theta)$.*
Let any one of the conditions 3.8(a) *and* 3.8(b) *of Theorem* 3.8 *be satisfied. Then θ will be a repelling point for A.*

We observe that it follows from the assumption $Ah_0 = \lambda_0 h_0$, $\lambda_0 > 1$, that 1 is not a positive characteristic value of the operator $A'(\theta)$ if this linear operator is h_0-bounded below (it suffices to apply Theorem 2.15).

Theorem 3.11. *If under the conditions of Theorem* 3.9 *the inequality $\lambda_0 < 1$ is replaced by the inequality $\lambda_0 > 1$, then the point θ will be repelling for the operator A.*

3.4. Inequalities for elements with large norms

3.4.1. *Existence of "progressing" and "retrogressing" elements*
In the previous section it was shown that the existence or absence of "progressing" and "retrogressing" elements with small norms for the operator A is determined basically by the properties of the derivative $A'(\theta)$. Analogously, in the investigation of the values of the operator A on elements with larger norms, it is convenient to make use of the derivative at infinity. Corresponding statements follow from the same arguments which were applied in section 3.3; therefore, we shall not repeat these arguments.

Theorem 3.12. *Let the operator A, positive on K, have a strong derivative $A'(\infty)$ at infinity with respect to the cone K where*

$$A'(\infty)g_0 = \lambda_\infty g_0 .$$

Let any one of the following two conditions be satisfied:
3.12(a) *The cone K is solid; g_0 is an interior element of K*;
3.12(b)

$$\lim_{t\to\infty} \frac{\|A(tg_0) - tA'(\infty)g_0\|_{g_0}}{t} = 0 .$$

Then a $t_0 = t_0(\lambda_\infty)$ can be found such that for $\lambda_\infty > 1$ for all $t \geqslant t_0$ the inequality

$$A(tg_0) \geqslant tg_0$$

will be satisfied and for $\lambda_\infty < 1$ for all $t \geqslant t_0$ the inequality

$A(tg_0) \leqslant tg_0$

will be satisfied.

In the application of this theorem, as also the subsequent theorems of the section, it is often convenient to make use of minorants or majorants of the operator being studied.

3.4.2. *Absence of "progressing" and "retrogressing" elements*

We mention two theorems. We shall call the operator A *g_0-bounded above* on the set T if for every $x \in T$ a $t > 0$ can be found such that $Ax \leqslant tg_0$. If g_0 is an interior element of the cone, then it is obvious that every operator is g_0-bounded above.

We call the operator A *uniformly g_0-bounded below at infinity* if to every $R > 0$ there corresponds a $t = t(R)$, such that from $x \in K$ and $\|x\| \geqslant R$ follows the relation $Ax \geqslant t(R)g_0$, where

$$\lim_{R \to \infty} t(R) = \infty .$$

We denote by $K(> R)$ the set of elements $x \in K$ such that $\|x\| > R$.

Theorem 3.13. *Let the positive operator A which is monotonic on some $K(> R)$ have a strong asymptotic derivative $A'(\infty)$ with respect to the cone K where*

$$A'(\infty)g_0 = \lambda_\infty g_0 \qquad (\lambda_\infty \geqslant 0; \quad g_0 \in K, \ g_0 \neq \theta) ,$$

and let the operator $A'(\infty)$ not have characteristic vectors in the cone K to which the characteristic value equal to 1 corresponds.

Let any one of the conditions 3.12(a), 3.12(b) *and any one of the following conditions be satisfied*:

3.13(a) *The operator A is completely continuous; the cone K normal.*

3.13(b) *The space E is weakly complete; the unit sphere in E is weakly compact; the operator A is weakly continuous, the cone K allows plastering.*

Then the following two statements are valid:

(1) *If A is g_0-bounded above, then from $\lambda_\infty < 1$ follows the existence of a number $R = R(\lambda_\infty)$ such that*

$$Ax \overline{\geqslant} x \qquad (x \in K, \quad \|x\| \geqslant R) .$$

(2) *If A is a uniformly g_0-bounded below at infinity then from $\lambda_\infty > 1$ follows the existence of a number $R = R(\lambda_\infty)$ such that*

$$Ax \overline{\leqslant} x \qquad (x \in K, \quad \|x\| \leqslant R) .$$

The proof of this and the previous theorem differs from the proofs of the corresponding statements of the previous section only in that instead of subsequences converging to zero sequences of elements the norms of which increase indefinitely must be considered.

The assumption concerning the satisfaction of one of the conditions 3.12(a) and 3.12(b) can be replaced by the assumption that

$$A(\varepsilon g_0) \leqslant \varepsilon g_0$$

for sufficiently large numbers $\varepsilon > 0$ if $\lambda_\infty < 1$ and that

$$A(\varepsilon g_0) \geqslant \varepsilon g_0$$

for sufficiently large ε if $\lambda_\infty > 1$.

We assume that the operator $A'(\infty)$ is a K-decomposable operator. This means (see p. 117) that every element $x \in E$ is uniquely representable in the form

$$x = f(x)h_0 + x_1 \qquad (x_1 \in E_1), \tag{3.52}$$

where $f(x)$ is a linear functional satisfying the conditions $f(h_0) = 1$ and $f(x_1) = 0$ for all x_1 of a subspace $E_1 \subset E$ which is invariant with respect to $A'(\infty)$ and having only zero as a common point with K; h_0 is a characteristic vector of the operator $A'(\infty)$ and $h_0 \in K$; the operator $A'(\infty)$ is defined by the equality

$$A'(\infty)x = \lambda_\infty f(x)h_0 + A'(\infty)x_1 . \tag{3.53}$$

The following statement is analogous to Theorems 3.9 and 3.11.

Theorem 3.14. *Let the operator A, defined on some $K(> R)$, have a strong asymptotic derivative with respect to the cone K which is a K-decomposable operator* (3.53). *Let the functional $f(x)$ appearing in the representation* (3.52) *be uniformly positive.*

Then a number $R = R(\lambda_\infty)$ exists such that for $\lambda_\infty < 1$

$$Ax \overline{\geqslant} x \qquad (x \in K, \quad \|x\| \geqslant R),$$

and for $\lambda_\infty > 1$

$$Ax \overline{\leqslant} x \qquad (x \in K, \quad \|x\| \geqslant R).$$

Chapter 4

EXISTENCE OF POSITIVE SOLUTIONS

Conditions are studied in this chapter for which the equation

$$x = Ax$$

with a positive non-linear operator A has at least one non-zero solution x^* in the cone K. It is assumed, in the majority of the cases considered below, that $A\theta = \theta$; thus, it is a question of a second (different from θ) solution in the cone K.

In the case of an equation in a one-dimensional space it is a question of the existence of positive numbers t^* such that $t^* = f(t^*)$, where $f(t)$ is a continuous and non-negative function for $t \in [0, \infty)$. Here, it suffices for the existence of a positive solution that either small values t be found such that $f(t) > t$ and large values t such that $f(t) < t$ or small numbers t such that $f(t) < t$ and large numbers t such that $f(t) > t$. It turns out that this elementary idea carries over to positive non-linear operators acting in Banach spaces with a cone.

4.1. Equations with monotone operators

4.1.1. *Operators, leaving a conical segment invariant*

In the sequel, the set $\langle x_0, u_0 \rangle$ of the elements x which satisfy the inequalities

$$x_0 \leqslant x \leqslant u_0$$

will be called a *conical segment.*

Theorem 4.1. *Let an operator A, monotone on the segment $\langle x_0, u_0 \rangle$, transform $\langle x_0, u_0 \rangle$ into itself, that is*

$$Ax_0 \geqslant x_0 \tag{4.1}$$

and

$$Au_0 \leqslant u_0 . \tag{4.2}$$

Then for the existence on $\langle x_0, u_0 \rangle$ of at least one fixed point x^* for the opera-

tor A, it suffices that any one of the following four conditions be satisfied:
4.1(a) *The cone K is strongly minihedral;*
4.1(b) *The cone K is regular, the operator A is continuous;*
4.1(c) *The cone K is normal, the operator A is completely continuous;*
4.1(d) *The cone K is normal, the space E is weakly complete, the unit sphere in E is weakly compact, the operator A is weakly continuous.*

Proof. Let condition 4.1(a) be satisfied. We denote by $\mathfrak{M}$ the set of points $x \in \langle x_0, u_0 \rangle$ such that $Ax \geqslant x$; this set is not empty since $x_0 \in \mathfrak{M}$. It follows from the monotoneity of the operator A that

$$A(Ax) \geqslant Ax \qquad (x \in \mathfrak{M}),$$

that is, A transforms $\mathfrak{M}$ into itself: $A\mathfrak{M} \subset \mathfrak{M}$. Let

$$x^* = \sup \mathfrak{M}.$$

It is obvious that $x_0 \leqslant x^* \leqslant u_0$, that is $x^* \in \langle x_0, u_0)$. It follows from the monotoneity of A that

$$Ax^* \geqslant Ax \geqslant x \qquad (x \in \mathfrak{M}).$$

This means that the element Ax^* is the least upper bound for $\mathfrak{M}$. Therefore

$$Ax^* \geqslant x^*.$$

But then $x^* \in \mathfrak{M}$, by virtue of which $Ax^* \in \mathfrak{M}$ and

$$Ax^* \leqslant x^*.$$

Thus, $Ax^* = x^*$.
Let condition 4.1(b) be satisfied. In this case, we form the sequence

$$x_n = Ax_{n-1} \qquad (n = 1, 2, \ldots). \tag{4.3}$$

It follows from condition (4.1) and from the monotoneity of the operator A that the sequence (4.3) is monotone and bounded:

$$x_0 \leqslant x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n \leqslant \ldots,$$
$$x_n \leqslant u_0 \qquad (n = 0, 1, 2, \ldots).$$

Since the cone K is regular, the sequence (4.3) converges in norm to some element $x^* \in \langle x_0, u_0 \rangle$. Passing in (4.3) to the limit as $n \to 0$, we obtain the equality $Ax^* = x^*$.
If condition 4.1(c) or 4.1(d) is satisfied, then the statement of the theorem follows respectively from Schauder's or Tikhonov's fixed point principle. This completes the proof of the theorem.

We observe that with the fulfillment of condition 4.1(c) or condition 4.1(d)

the fixed point x^* of the operator A also can be obtained as the limit of the sequence (4.3); for condition (4.1c) convergence follows from its compactness and for condition 4.1(d) the sequence (4.3) converges weakly to x^*.

4.1.2. *Possibility of the existence of several fixed points*
The conditions of Theorem 4.1 do not guarantee the uniqueness of the fixed point for the operator A on a conical segment $\langle x_0, u_0\rangle$. We mention a corresponding example.
Let E be a one-dimensional space of real numbers, and K be a set of non-negative numbers. Let

$$Ax = x + \frac{\sin x}{2} \qquad (x \geqslant 0)\,.$$

This operator (this is a function) is monotone since the derivative is positive. We set

$$x_0 = \frac{\pi}{2}, \quad u_0 = \frac{7\pi}{2}\,.$$

Then the inequalities (4.1) and (4.2) will be satisfied. At the same time the operator A on $[(\pi/2), (7\pi/2)]$ has three fixed points: π, 2π, 3π.

4.1.3. *Determination of an invariant conical segment*
Let the operator A be defined on all the cone K and be monotone on it. Then for the construction of a conical segment $\langle x_0, u_0\rangle$ which is invariant for the operator A, it suffices to set $x = \theta$ and to find the element u_0, which satisfies the inequality (4.2). In the determination of the element u_0 we can make use of the Theorem 3.11.
In this manner, we can obtain, for example, the following statement:

Let the cone K be solid and the operator A positive on K (continuous if K is regular and completely continuous if K is normal) have a strong derivative $A'(\infty)$ with respect to a cone with the characteristic vector g_0 being an interior element of K. Then for the existence of a fixed point for the operator A in the cone K, it suffices that a characteristic value less than 1 *correspond to the characteristic vector g_0.*

If $A\theta = \theta$ then, for the proof of the existence of non-zero fixed points for the operator A in the cone K, we can also use Theorem 4.1. Here, we attempt, with the help of methods studied in section 3 of the preceding chapter, to find an element with a small norm x_0 satisfying the condition

$Ax \geqslant x_0$ and then, by means of methods from section 4 of the last chapter, to look for an element u_0 satisfying the condition $Au_0 \leqslant u_0$. If it turns out that $x_0 \leqslant u_0$ then, for suitable assumptions about the properties of the operator and the cone, the existence of a fixed point will be proved.
Thus, we can offer the following plan for the proof of the existence of a non-zero fixed point for the operator $A(A\theta = \theta)$ which is monotone on K. It is necessary to find the operators $A'(\theta)$ and $A'(\infty)$ and their characteristic vectors h_0 and g_0:

$$A'(\theta)h_0 = \lambda_0 h_0\,, \quad A'(\infty)g_0 = \lambda_\infty g_0\,.$$

If it turns out that

$$\lambda_\infty < 1 < \lambda_0$$

and for some t_0

$$h_0 \leqslant t_0 g_0\,,$$

then it follows that we must attempt to consider as an invariant segment $\langle x_0, u_0\rangle$ for A the conical segments $\langle \varepsilon_1 h_0, (1/\varepsilon_2)g_0\rangle$ for sufficiently small numbers ε_1, ε_2. If an invariant conical segment is found, then it follows that we must attempt to apply Theorem 4.1.
In the determination of invariant segments $\langle x_0, u_0\rangle$ one must also apply the minorants and majorants of the operator A.

4.1.4. *Special class of monotone operators*
Theorem 4.2. *Let the monotone and continuous operator A on the conical segment $\langle x_0, u_0\rangle$ satisfy the condition*

$$A(x+y) \geqslant Ax + \alpha(\|y\|)v_0 \qquad (x,\ x+y \in \langle x_0, u_0\rangle,\ y \in K)\,, \tag{4.4}$$

where $\alpha(r)$ is a non-decreasing continuous function which is positive for $r>0$ and v_0 is a fixed non-zero element of K.
Let A transform $\langle x_0, u_0\rangle$ into itself.
Then at least one fixed point of the operator A exists on $\langle x_0, u_0\rangle$.

Proof. We consider the sequence (4.3). If this sequence converges to some element x^*, then by passing to the limit in (4.3) we obtain the equality $Ax^* = x^*$ and the theorem is thereby proved.
We assume that the sequence (4.3) does not converge in norm. Then from it a subsequence

$$x_{n_0} \leqslant x_{n_1} \leqslant \ldots \leqslant x_{n_i} \leqslant \ldots \leqslant u_0$$

can be chosen such that

$$\|x_{n_{i+1}} - x_{n_i}\| \geqslant r_0 > 0 \qquad (i = 0, 1, 2, \ldots).$$

Let $\alpha(r_0) = \varepsilon_0$. Then, by virtue of (4.4),

$$x_{n_{i+1}} = Ax_{n_i} \geqslant Ax_{n_{i-1}} + \varepsilon_0 v_0 = x_{n_{i-1}+1} + \varepsilon_0 v_0,$$

from which

$$x_{n_{i+1}} \geqslant x_{n_i+1} \geqslant x_{n_{i-1}+1} + \varepsilon_0 v_0 \geqslant x_{n_{i-1}} + \varepsilon_0 v_0.$$

It follows from the last inequalities that

$$x_{n_2} \geqslant x_{n_0} + \varepsilon_0 v_0, \quad x_{n_4} \geqslant x_{n_0} + 2\varepsilon_0 v_0, \ldots,$$

that is,

$$u_0 \geqslant x_{n_{2i}} \geqslant x_{n_0} + i\varepsilon_0 v_0 \geqslant i\varepsilon_0 v_0 \qquad (i = 1, 2, \ldots).$$

But then the element $-v_0$ is the limit of the sequence of elements

$$\frac{u_0}{i\varepsilon_0} - v_0 \in K \qquad (i = 1, 2, \ldots),$$

that is $-v_0 \in K$. This means that $v_0 = \theta$. We have thus arrived at a contradiction.

This completes the proof of the theorem.

This theorem could be obtained as a corollary of Theorem 4.1 (for this it is necessary to choose a minimal subspace E_1, containing $\langle x_0, u_0\rangle$, and investigate the properties of the cone $K_1 = K \cap E_1$). We preferred a direct proof.

4.1.5. *Use of a second cone*

Let there be given in the space E two cones K_0 and K, where $K_0 \subset K$ (see chapter 1, section 1.8). We shall consider in the space E only the partial ordering generated by the larger cone K.

We denote by $K_0\langle x_0, u_0\rangle$ the intersection of the conical segment $\langle x_0, u_0\rangle$ with the cone K_0, that is, the collection of elements $x \in K_0$ such that $x_0 \leqslant x \leqslant u_0$. The statements proved in the present section transfer without difficulty to equations with operators leaving the sets $K_0\langle x_0, u_0\rangle$ invariant. We give one example.

Theorem 4.3. *Let the operator A, monotone on $K_0\langle x_0, u_0\rangle$, transform $K_0\langle x_0, u_0\rangle$ into itself.*

Then for the existence on $K_0\langle x_0, u_0\rangle$ of at least one fixed point for the operator A, it suffices that any one of the following conditions be satisfied:

4.3(a) *The cone K_0 is K-regular, the operator A is continuous;*

4.3(b) *The set* $K_0\langle x_0, u_0\rangle$ *is bounded**, *the operator* A *is completely continuous*;
4.3(c) *The set* $K_0\langle x_0, u_0\rangle$ *is bounded, the space* E *is weakly complete, the unit sphere in* E *is weakly compact, the operator* A *is weakly continuous.*

The proof of this theorem does not require any new considerations. Theorem 4.2 also generalizes without modifications.
In the determination of the elements $x_0, u_0 \in K_0$ satisfying the conditions $x_0 \leqslant Ax_0$, $Au_0 \leqslant u_0$, it is convenient to make use of the analogous Theorems 3.5 and 3.11. In these theorems it is obvious that it suffices to consider the derivatives with respect to the "smaller" cone K_0; in this connection, in conditions 3.5(a) and 3.11(a), it suffices to assume that the "larger" cone K is solid and that h_0 (corresponding to g_0) is an interior element of K.

4.1.6. *Remark about the convergence of successive approximations*
Lemma 4.1 (about two militia-men). *Let*

$$x_n \leqslant y_n \leqslant u_n \qquad (n=1, 2, \ldots) \tag{4.5}$$

and let the sequences x_n *and* u_n *converge in norm to one and the same element* x^*. *Let the cone* K *be normal.*
Then

$$\lim_{n\to\infty} \|y_n - x^*\| = 0 .$$

The proof is almost obvious: it follows from (4.5) that

$$\theta \leqslant y_n - x_n \leqslant u_n - x_n \qquad (n=1, 2, \ldots) ,$$

and since the cone K is normal, then, by virtue of Theorem 1.2,

$$\|y_n - x_n\| \leqslant L\|u_n - x_n\| \qquad (n=1, 2, \ldots) ,$$

from which it follows that

$$\|y_n - x^*\| \leqslant (L+1)\|x_n - x^*\| + L\|u_n - x^*\| \qquad (n=1, 2, \ldots) .$$

This completes the proof of the lemma.

Theorem 4.4. *Let the operator* A *be monotone on* $\langle x_0, u_0\rangle$ *and transform* $\langle x_0, u_0\rangle$ *into itself. Let either one of the conditions* 4.1(b), 4.1(c) *of Theorem* 4.1 *or the condition* (4.4) *of Theorem* 4.2 *be satisfied.*

* See Theorem 1.16.

Let the operator A have a unique fixed point x^* on $\langle x_0, u_0\rangle$. Then the successive approximations

$$y_n = Ay_{n-1} \qquad (n=1, 2, \ldots) \tag{4.6}$$

converge in norm to x^* no matter what $y_0 \in \langle x_0, u_0\rangle$ is.

Proof. We form the sequences

$$x_n = Ax_{n-1}, \quad u_n = Au_{n-1} \qquad (n=1, 2, \ldots). \tag{4.7}$$

These sequences and the sequence (4.6) obviously satisfy the inequalities (4.5).

Each of the sequences (4.7) is monotone and bounded. For condition 4.1(c) both sequences converge since they are monotone and compact. The convergence of the sequence x_n under the conditions of the Theorem 4.2 has been proved; the convergence of the sequence u_n is established analogously: assuming the contrary, we can construct a subsequence

$$u_{n_0} \geqslant u_{n_1} \geqslant u_{n_2} \geqslant \ldots \geqslant u_{n_i} \geqslant \ldots \geqslant x_0$$

such that

$$\|u_{n_i} - u_{n_{i+1}}\| \geqslant r_0 > 0 \qquad (i=0, 1, 2, \ldots);$$

then

$$u_{n_0} \geqslant Au_{n_0} \geqslant Au_{n_1} + \alpha(r_0)v_0 = u_{n_1+1} + \alpha(r_0)v_0 \geqslant u_{n_2} + \alpha(r_0)v_0$$

and analogously

$$u_{n_{2i}} \geqslant u_{n_{2(i+1)}} + \alpha(r_0)v_0 \qquad (i=0, 1, 2, \ldots);$$

therefore

$$u_0 \geqslant u_{n_0} \geqslant u_{n_2} + \alpha(r_0)v_0 \geqslant u_{n_4} + 2\alpha(r_0)v_0 \geqslant \ldots \geqslant u_{n_{2i}} + i\alpha(r_0)v_0 \geqslant i\alpha(r_0)v_0 \qquad (i=1, 2, \ldots)$$

and since the number i can be chosen as large as desired, then $-v_0 \in K$, which contradicts the assumption.

Therefore, both sequences (4.7) converge. The limits of these sequences are fixed points of the operator A. It follows from the uniqueness of the fixed point that the limits of the sequences (4.7) coincide. Lemma 4.1 about two militiamen remains to be applied.

This completes the proof of the theorem.

4.1.7. *Equations in spaces with fully regular cones*

In the preceding subsections the solution of the equations $x = Ax$ with a monotone operator A was constructed in several cases as the limit of a

monotone sequence

$$x_n = Ax_{n-1} \qquad (n=1, 2, \ldots). \tag{4.8}$$

In this connection the sequence (4.8) is shown to be bounded in the sense of a partial ordering. If the cone K is fully regular then it suffices for the convergence of sequence (4.8) that there be boundedness in norm if this sequence is monotone. This consideration reduces to a series of theorems about the existence of positive solutions to the equation $x=Ax$. We mention one of them.

Theorem 4.5. *Let the operator A, positive, monotone and continuous on a cone, have a strong asymptotic derivative $A'(\infty)$ with respect to the cone. Let the spectrum of the linear operator $A'(\infty)$ lie in the circle of radius ϱ_0 where $\varrho_0<1$.*
Let the cone K be fully regular.
Then the operator A has at least one fixed point on K.

Proof. By virtue of the Lemma 2.2 a norm $\|x\|_0$ can be defined in E such that

$$m\|x\| \leqslant \|x\|_0 \leqslant M\|x\| \qquad (x\in E),$$

where m, $M>0$ and

$$\|A'(\infty)\|_0 = \sup_{\|x\|\leqslant 1} \|A'(\infty)x\|_0 = 1-\eta,$$

where $\eta>0$.
We choose a number $R>0$ such that for $\|x\| \geqslant R (x\in K)$

$$\frac{\|Ax-A'(\infty)x\|}{\|x\|} < \frac{m}{M}\eta.$$

Then for $\|x\| \geqslant R (x\in K)$

$$\|Ax\|_0 \leqslant \|Ax-A'(\infty)x\|_0 + \|A'(\infty)x\|_0 \leqslant$$
$$\leqslant M\|Ax-A'(\infty)x\| + (1-\eta)\|x\|_0,$$

that is

$$\|Ax\|_0 < m\eta\|x\| + (1-\eta)\|x\|_0 \leqslant \|x\|_0 \qquad (\|x\| \geqslant R,\ x\in K).$$

We set $x_0=\theta$ and $x_n=Ax_{n-1}$ $(n=1, 2, \ldots)$. The constructed sequence is monotone and, by virtue of the last inequality, bounded in norm. It follows from the full regularity of the cone K that the sequence x_n converges in norm. Its limit is a fixed point of the operator A.
This completes the proof of the theorem.

This same argument scheme is applicable in the proof of the existence of non-zero positive solutions, if $A\theta=\theta$.

Theorem 4.6. *Let the conditions of the Theorem* 4.5 *be satisfied. Let a non-zero element* $x_0\in K$ *exist such that* $Ax_0\geqslant x_0$.
Then the operator A *has at least one non-zero fixed point on* K.

If the operator A is completely continuous, then the statements of Theorems 4.5 and 4.6 remain valid if the proposition about the full regularity of the cone K is discarded. For a completely continuous linear operator $A'(\infty)$ the assumption that the spectrum lies in the circle of radius $\varrho_0<1$ is equivalent, generally speaking (see chapter 2), to the assumption that $A'(\infty)$ does not have positive characteristic values λ_0 greater than or equal to 1.

4.2. Equations with non-monotone operators

4.2.1. *Existence of non-negative solutions*
For the proof of the existence of fixed points for non-monotone positive operators A, it is convenient to apply different general fixed point principles and, in particular, the Schauder and Tikhonov principles. We postulate for the operator which is being considered, complete continuity or (if the space E is weakly complete and the unit sphere weakly compact) weak continuity. The proof of the existence of a fixed point reduces to the determination of a convex set $T\subset E$ which is left invariant by the operator A. The conical segments $\langle x_0, u_0\rangle$, the intersections $K_0\langle x_0, u_0\rangle$ of the conical segment $\langle x_0, u_0\rangle$ with some cone $K_0\subset K$, the intersections $K(r)$ of the cone K with the sphere $\|x\|\leqslant r$ and so on, can appear as the set T. We mention a few propositions obtained in this manner.

Theorem 4.7. *Let the operator* A, *positive on the cone* K, *have a strong asymptotic derivative* $A'(\infty)$ *respect with to a cone where the spectrum of the linear operator* $A'(\infty)$ *lies in the circle* $|\lambda|\leqslant\varrho_0$, *where* $\varrho_0<1$.
Then it suffices for the existence in the cone K *of at least one fixed point for the operator* A *that any one of the following conditions be satisfied*:
4.7(a) *The operator* A *is completely continuous.*
4.7(b) *The space* E *is weakly complete, the unit sphere is weakly compact, the operator* A *is weakly continuous.*

Proof. By virtue of the Lemma 2.2 there can be introduced in E a norm $\|x\|_0$ equivalent to the originally given one such that

$$\|A'(\infty)\|_0 = 1-\eta,$$

where $\eta>0$. In the proof of the Theorem 4.5, it was established that in the new norm the operator A satisfies the inequalities (4.8)

$$\|Ax\|_0 < \|x\|_0 \qquad (x\in K,\ \|x\|_0 \geqslant R),$$

where R is some positive number. This means that the operator A transforms into itself the intersection T of the cone K with the sphere $\|x\|_0 \leqslant R_0$, where

$$R_0 = \max\left\{R, \sup_{x\in K,\ \|x\|_0\leqslant R} \|Ax\|_0\right\}.$$

The set T is bounded, convex and closed. The existence for the operator A of a fixed point $x^*\in T$ follows from the Schauder principle for condition 4.7(a) and from Tikhonov's principle for condition 4.7(b).
This completes the proof of the theorem.

Theorem 4.8. *Let the positive operator A have on the cone K the majorant A^+:*

$$A^+x \geqslant Ax \qquad (x\in K), \tag{4.9}$$

where the majorant A^+ has a strong asymptotic derivative $A^+(\infty)$ with respect to a cone with the spectrum lying in the circle $|\lambda|\leqslant\varrho_0$, where $\varrho_0<1$.
Let the cone K be normal and, moreover, the norm be monotone.
Let one of the conditions 4.7(a) *and* 4.7(b) *of the preceding theorem be satisfied.*
Then the operator A has at least one fixed point $x^*\in K$.

Proof. As in the proof of Theorem 4.7 we introduce a norm $\|x\|_0$ such that for some number R

$$\|A^+x\|_0 \leqslant \|x\|_0 \qquad (x\in K,\ \|x\|_0\geqslant R).$$

By virtue of Lemma 2.3 a new norm can be introduced such that it retains the property of monotoneity: the inequality $\|x\|_0\leqslant\|y\|_0$ follows from $\theta\leqslant x\leqslant y$. Therefore, it follows from (4.9) that

$$\|Ax\|_0 \leqslant \|A^+x\|_0 \leqslant \|x\|_0 \qquad (x\in K,\ \|x\|_0\geqslant R).$$

Further arguments are as in the proof of Theorem 4.7.
This completes the proof of the theorem.

If in the conditions of Theorem 4.8 the operator A leaves the cone $K_0\subset K$

invariant, then it suffices that the majorant A^+ be defined only on the cone K_0.

4.2.2. *Overdetermination of the operator*

In the proof of the existence of fixed points, the following simple consideration often happens to be convenient. The operator A, studied on a part of its domain of definition, is overdetermined in order that the existence of a fixed point for the overdetermined operator is obvious. Thus, it is proved that a fixed point cannot be found where the values of the operator A were overdetermined. This simple idea is extraordinarily fruitful (in the sequel it will be applied repeatedly): the main difficulty in its realization is the regular overdetermination of the operator A.

We mention an example of the statement which is near to Theorem 4.8 but does not require the monotoneity of the norm. We restrict ourselves to the cases of a completely continuous operator.

Theorem 4.9. *Let a number R exist for a positive completely continuous operator A such that for all $\varepsilon>0$*

$$Ax \geqq (1+\varepsilon)x \qquad (x\in K,\ \|x\|=R)\,. \tag{4.10}$$

Then the operator A has at least one fixed point $x^*\in K$.

Proof. We define the operator $\tilde{A}$ by the equality

$$\tilde{A}x = \begin{cases} Ax, & \text{if } x\in K,\ \|x\|\leqslant R\,, \\ A\left(\dfrac{R}{\|x\|}\,x\right), & \text{if } x\in K,\ \|x\|\geqslant R\,. \end{cases}$$

The operator $\tilde{A}$ is completely continuous and transforms the cone K into a compact subset of K. By virtue of the Schauder principle, an element $x^*\in K$ can be found such that $\tilde{A}x^*=x^*$.

We assume that $\|x^*\|>R$ and set

$$y^* = \frac{R}{\|x^*\|}\,x^*\,.$$

Then

$$Ay^*=\tilde{A}y^* = \frac{\|x^*\|}{R}\,y^*\,,$$

which contradicts (4.10).

Thus, $\|x^*\|\leqslant R$. Therefore, $Ax^*=\tilde{A}x^*=x^*$.

This completes the proof of the theorem.

We remark that for the examination of condition (4.10) it is convenient to consider various majorants of the operator A. It is not difficult to see that condition (4.10) is satisfied if for some majorant A^+ the relations

$$A^+x \gtrsim (1+\varepsilon)x \qquad (x \in K,\ \|x\| = R)$$

are valid or if the more severe restriction

$$A^+x \gtrsim x \qquad (x \in K,\ \|x\| = R) \tag{4.11}$$

is satisfied.

The question about when condition (4.11) is satisfied was studied in chapter 3 (section 3.4, subsection 3.4.2). We make use of the results obtained there and formulate one of the corollaries of Theorem 4.9 in the form of a separate theorem which is convenient in the applications.

Theorem 4.9′. *Let a completely continuous positive operator A have on K the majorant A^+ which is a monotone completely continuous operator. Let a strong asymptotic derivative $A^{+\prime}(\infty)$ exist with respect to a cone having a positive characteristic vector g_0:*

$$A^{+\prime}(\infty)g_0 = \lambda_\infty g_0\,,$$

where $\lambda_\infty < 1$ and 1 is not a positive characteristic value of the operator $A^{+\prime}(\infty)$.

Let the operator A^+ be g_0-bounded above. Let the cone K be normal.

Then the operator A has at least one fixed point on K.

We mention one more statement concerning the existence of a positive solution $x^* \in K$; the overdetermination of the operator A is used in an essential way in the proof.

Theorem 4.10. *We assume that the positive operator A is completely continuous and has a strong asymptotic derivative $A'(\infty)$ with respect to a cone. Let the linear operator $A'(\infty)$ not have positive characteristic values exceeding or equal to 1. Then A has a fixed point on K.*

Proof. For the proof we construct the sequence of operators

$$A_n x = \begin{cases} Ax\,, & \text{if} \quad \|x\| \leqslant n\,, \\ \left(2 - \dfrac{\|x\|}{n}\right) A\left(\dfrac{n}{\|x\|}\,x\right), & \text{if} \quad n \leqslant \|x\| \leqslant 2n\,, \\ \theta\,, & \text{if} \quad \|x\| \geqslant 2n\,. \end{cases} \tag{4.12}$$

Each of the operators A_n evidently is completely continuous and transforms the entire cone K into its compact part. It follows from the Schau-

der principle that every operator A_n has at least one fixed point x_n. It is obvious that

$$\|x_n\| < 2n \qquad (n=1, 2, \ldots).$$

Therefore, the proof will be complete if it is proved that for large num bers n the inequalities $\|x_n\| > n$ are not satisfied.
We assume the contrary. Let the equalities

$$\left(2 - \frac{\|x_n\|}{n}\right) A\left(\frac{n}{\|x_n\|} x_n\right) = x_n \qquad (n=1, 2, \ldots),$$

which we rewrite in the form

$$\alpha_n \frac{Ay_n - A'(\infty)y_n}{\|y_n\|} + \alpha_n A'(\infty) \frac{y_n}{\|y_n\|} = \frac{y_n}{\|y_n\|} \qquad (n=1, 2, \ldots),$$

where

$$y_n = \frac{n}{\|x_n\|} x_n, \quad \alpha_n = 2\frac{n}{\|x_n\|} - 1 \qquad (n=1, 2, \ldots),$$

and let

$$n < \|x_n\| < 2n \qquad (n=1, 2, \ldots).$$

Without loss of generality we can assume that the numbers α_n converge to some limit α_0, where it follows from the inequalities $0<\alpha_n<1$ that $0 \leqslant \alpha_0 \leqslant 1$. Simultaneously, we shall assume that the elements

$$A'(\infty) \frac{y_n}{\|y_n\|} \qquad (n=1, 2, \ldots)$$

converge in norm to some element v_0. Then the elements $y_n/\|y_n\|$ will converge to the element $\alpha_0 v_0 = u_0$. Consequently, $\alpha_0 \neq 0$ and the equality

$$u_0 = \alpha_0 A'(\infty) u_0$$

holds which contradicts the assumption which states that the operator $A'(\infty)$ does not have positive characteristic values greater than 1 or equal to 1. With this the proof is complete.

In the last statement the assumption about complete continuity of the operator A can be replaced by the assumption of weak continuity if the space E is weakly complete, if the unit sphere is weakly compact and if the cone K allows plastering. The proof in this connection almost does not change. It is necessary only that the operators A_n be defined on the cone K by the equalities

$$A_n x = \begin{cases} Ax, & \text{if } l(x) \leqslant n, \\ \left[2 - \frac{l(x)}{n}\right] A\left[\frac{n}{l(x)} x\right], & \text{if } n \leqslant l(x) \leqslant 2n, \\ \theta, & \text{if } l(x) \geqslant 2n \end{cases}$$

instead of by the formulas (4.12) and thus the Tikhonov principle be applied instead of the Schauder principle.
We observe that the last statements contain Theorem 4.7 as a particular case.

4.2.3. *Positive solutions*
We assume that $A\theta = \theta$ and raise the question concerning the existence in the cone of other (different from θ) fixed points for the positive operator A.

Theorem 4.11. *Let the positive operator $A(A\theta = \theta)$ have a strong Fréchet derivative $A'(\theta)$ and a strong asymptotic derivative $A'(\infty)$ with respect to a cone. Let the spectrum of the operator $A'(\infty)$ lie in the circle $|\lambda| \leqslant \varrho$, where $\varrho < 1$. Let the operator $A'(\theta)$ have in K a characteristic vector h_0; then*

$$A'(\theta) h_0 = \lambda_0 h_0,$$

where $\lambda_0 > 1$ and $A'(\theta)$ does not have in K characteristic vectors to which a characteristic value equal to 1 corresponds.
Then it suffices for the existence for the operator A of at one non-zero fixed point in the cone K that any one of the following conditions be satisfied:
4.11(a) *The operator A is completely continuous.*
4.11(b) *The operator A is weakly continuous, the space E is weakly complete, the unit sphere in E is weakly compact, the cone K allows plastering.*

Proof. Let, to begin with, the condition 4.11(a) be satisfied. Then we define the operators A_r by the equality

$$A_r x = \begin{cases} Ax, & \text{if } x \in K, \ \|x\| \geqslant r, \\ Ax + (r - \|x\|) h_0, & \text{if } x \in K, \ \|x\| \leqslant r. \end{cases}$$

The complete continuity of all the operators A_r is obvious. They satisfy all the conditions of Theorem 4.7 and have, by virtue of this, fixed points in the cone K. We shall show that the norms of these fixed points are greater than r if r is sufficiently small. Then the fixed points of the operator A_r will be non-zero fixed points of the operator A and the theorem will be proved.
We assume that every operator A_r has a fixed point whose norm does not

exceed r. Then we can construct a sequence $x_n \in K$ such that

$$0 < \|x_n\| \leqslant \frac{1}{n} \qquad (n=1, 2, \ldots) \tag{4.13}$$

and

$$Ax_n + \left(\frac{1}{n} - \|x_n\|\right)h_0 = x_n \qquad (n=1, 2, \ldots). \tag{4.14}$$

Without loss of generality, we can assume that the sequence

$$A'(0)\frac{x_n}{\|x_n\|} \qquad (n=1, 2, \ldots)$$

converges in norm to some element $z \in K$.
We rewrite the equality (4.14) in the form

$$\left(\frac{1}{n\|x_n\|} - 1\right)h_0 = \frac{x_n}{\|x_n\|} - \frac{Ax_n - A'(\theta)x_n}{\|x_n\|} - A'(\theta)\frac{x_n}{\|x_n\|}. \tag{4.15}$$

It follows from this equality that

$$\|h_0\| \overline{\lim_{n\to\infty}} \left(\frac{1}{n\|x_n\|} - 1\right) \leqslant 1 + \|z\|.$$

Therefore, without loss of generality, we can assume that the sequence $1/(n\|x_n\|) - 1$ converges to some number α which, by virtue of (4.13), is non-negative. It then follows from the equalities (4.15) that the sequence $x_n/\|x_n\|$ $(n=1, 2, \ldots)$ converges in norm to some vector $u_0 \in K$ where $\|u_0\| = 1$. Passing to the limit in (4.15) we obtain the equality

$$u_0 = A'(\theta)u_0 + \alpha h_0. \tag{4.16}$$

The number α is positive since $A'(\theta)$ does not have characteristic vectors in K to which there corresponds a characteristic value equal to unity. Consequently, a number $t_0 > 0$ can be found such that $u_0 \geqslant t_0 h_0$ and from $u \geqslant th_0$ follows the inequality $t \leqslant t_0$. At the same time, the inequality

$$u_0 \geqslant A'(\theta)(t_0 h_0) + \alpha h_0 = (\lambda_0 t_0 + \alpha)h_0,$$

follows from (4.16), that is, $u_0 \geqslant th_0$, where $t = \lambda_0 t_0 + \alpha > t_0$. We have arrived at a contradiction.
This completes the proof of the theorem for condition 4.11(a).

Let condition 4.11(b) be satisfied. We denote by $l(x)$ a uniformly positive linear functional on K. We set

$$A_r x = \begin{cases} Ax, & \text{if } x \in K,\ l(x) \geqslant r, \\ Ax + [r - l(x)]h_0, & \text{if } x \in K,\ l(x) \leqslant r. \end{cases}$$

The operators A_r evidently are weakly continuous and, by virtue of the Theorem 4.7, have fixed points on K. As under the fulfillment of condition 4.11(a), it is necessary only to show that for sufficiently small numbers r the norms of these fixed points are greater than r.
Assuming the contrary, we find that a sequence x_n exists such that

$$0<l(x_n)\leqslant\frac{1}{n}\qquad(n=1,2,\ldots)$$

and

$$Ax_n+\left[\frac{1}{n}-l(x_n)\right]h_0=x_0\qquad(n=1,2,\ldots),$$

that is

$$\left[\frac{1}{nl(x_n)}-1\right]h_0=\frac{x_n}{l(x_n)}-\frac{Ax_n-A'(\theta)x_n}{\|x_n\|}\cdot\frac{\|x_n\|}{l(x_n)}-A'(\theta)\frac{x_n}{l(x_n)}. \tag{4.17}$$

Without loss of generality, we can assume that the sequence $x_n/l(x_n)$ converges weakly to some element $u_0\in K$; in this connection $l(u_0)=1$. Then the sequence $A'(\theta)[x_n/l(x_n)]$ converges weakly to the element $A'(\theta)u_0$. By virtue of (4.17) the numbers $[1/nl(x_n)]-1$ converge to the number

$$\alpha=\frac{l(u_0)-l[A'(\theta)u_0]}{l(h_0)}$$

[in order to be convinced of this, it suffices to apply the functional $l(x)$ to the equality (4.17)]. Therefore, the equality (4.16) follows from (4.17) which, as we already know, brings us to a contradiction.

4.2.4. *Fixed point in a compressed cone*
We say that the operator $A(A\theta=\theta)$ is a *compression of the cone* if numbers $r, R>0$ can be found such that

$$Ax\overline{\leqslant}x\qquad(x\in K,\ \|x\|\leqslant r,\ x\neq\theta), \tag{4.18}$$

and for all $\varepsilon>0$

$$Ax\overline{\geqslant}(1+\varepsilon)x\qquad(x\in K,\ \|x\|\geqslant R). \tag{4.19}$$

Theorem 4.12. *Let the positive completely continuous operator A be a compression of the cone.*
Then the operator A has at least one non-zero fixed point on K.

Proof. Without loss of generality, we can assume that

$$0<r<1<R<\infty.$$

We choose an element $h_0 \in K$ such that

$$\|h_0\| > \frac{1}{1-r}\left(r^2 + r \sup_{y \in K,\, \|y\| = r} \|Ay\|\right). \tag{4.20}$$

We define the operator $\tilde{A}$ on K by the equality

$$\tilde{A}x = \begin{cases} \dfrac{\|x\|}{r} A\left(\dfrac{r}{\|x\|} x\right) + (1-r)h_0, & \text{if } \|x\| \leqslant r^2, \\ \dfrac{\|x\|}{r} A\left(\dfrac{r}{\|x\|} x\right) + \dfrac{\|x\|}{r} \cdot \dfrac{r - \|x\|}{\|x\|} h_0, & \text{if } r^2 \leqslant \|x\| \leqslant r, \\ Ax, & \text{if } r \leqslant \|x\| \leqslant R, \\ A\left(\dfrac{R}{\|x\|} x\right), & \text{if } \|x\| \geqslant R. \end{cases} \tag{4.21}$$

It is not difficult to see that the operator $\tilde{A}$ is completely continuous and transforms the cone K into its compact part. By virtue of the Schauder principle, the operator $\tilde{A}$ has on K a fixed point x^*. For the completion of the proof, it suffices to show that

$$r \leqslant \|x^*\| \leqslant R.$$

We assume that $\|x^*\| \leqslant r^2$. Then

$$x^* = \frac{\|x^*\|}{r} A\left(\frac{r}{\|x^*\|} x^*\right) + (1-r)h_0,$$

from which

$$\|h_0\| \leqslant \frac{1}{1-r}\left(r^2 + r\left\|A\left(\frac{r}{\|x^*\|} x^*\right)\right\|\right),$$

which contradicts (4.20).

We assume that $r^2 < \|x^*\| < r$. Then

$$\frac{r}{\|x^*\|} x^* = A\left(\frac{r}{\|x^*\|} x^*\right) + \frac{r - \|x^*\|}{\|x^*\|} h_0,$$

from which

$$\frac{r}{\|x^*\|} x^* \geqslant A\left(\frac{r}{\|x^*\|} x^*\right),$$

which contradicts (4.18).

Finally, we assume that $\|x^*\| > R$. Then

$$x^* = A\left(\frac{R}{\|x^*\|} x^*\right),$$

that is

$$A\left(\frac{R}{\|x^*\|}x^*\right)=(1+\varepsilon)\cdot\frac{R}{\|x^*\|}x^*,$$

where

$$\varepsilon=\frac{\|x^*\|}{R}-1.$$

Again, we have arrived at a contradiction.
This completes the proof of the theorem.

Theorem 4.13. *Let the space E be weakly complete and the unit sphere in E weakly compact. Let the cone K allow plastering.*
Let the positive weakly continuous operator A be a compression of the cone. Then the operator A has at least one non-zero fixed point on K.

The proof is carried out analogously to the proof of the preceding theorem with the change that the operator $\tilde{A}$ must be defined by the formula

$$\tilde{A}x=\begin{cases}\dfrac{l(x)}{r_0}A\left[\dfrac{r_0}{l(x)}x\right]+(1-r_0)h_0, & \text{if } l(x)\leqslant r_0^2,\\ \dfrac{l(x)}{r_0}A\left[\dfrac{r_0}{l(x)}x\right]+\dfrac{l(x)}{r_0}\dfrac{r_0-l(x)}{l(x)}h_0, & \text{if } r_0^2\leqslant l(x)\leqslant r_0,\\ Ax, & \text{if } r_0\leqslant l(x)\leqslant R_0,\\ A\left[\dfrac{R_0}{l(x)}x\right], & \text{if } l(x)\geqslant R_0,\end{cases}\tag{4.22}$$

where $l(x)$ is a uniformly positive linear functional and r_0 and R_0 are, respectively, sufficiently small and sufficiently large positive numbers. It is necessary to use Tikhonov's principle instead of Schauder's.
It is seen from the proof that in Theorems 4.12 and 4.13 the condition (4.18) can be replaced by the assumption about the existence of a non-zero $h_0\in K$ such that for $\alpha>0$ and $x\in K$ $(0<\|x\|\leqslant r)$ the inequality

$$x\neq Ax+\alpha h_0$$

is satisfied. Condition (4.19) can be replaced by the assumption that

$$Ax\neq(1+\varepsilon)x \qquad (x\in K,\ \|x\|\geqslant R,\ \varepsilon>0).$$

These assumptions are less restricted but are not convenient since their examination does not allow the use of minorants and majorants.

4.2.5. *Use of minorants and majorants*

The application of the last two theorems requires the examination of conditions (4.18) and (4.19) Here, theorems from chapter 3 (sections 3.3 and

3.4) can be applied.

It is convenient to use the minorants A^- and the majorants A^+: if the condition (4.18) is satisfied for the minorant A^-

$$A^-x \lesseqgtr x \qquad (x \in K,\ 0 < \|x\| \leqslant r), \tag{4.23}$$

then it is all the more satisfied for the operator A: analogously, (4.19) follows from

$$A^+x \gtreqless x \qquad (x \in K,\ \|x\| \geqslant R) \tag{4.24}$$

It is necessary to construct the minorant (4.23) only on the elements having a small norm, and the majorants, on elements with a large norm.

The linear operators can be used as minorants and majorants. Then, in the examination of conditions (4.23) and (4.24), Theorems 2.17 and 2.18 can be applied.

One further remark. In the theorems mentioned above the proofs were based on the overdetermination of the operator A on the elements $x \in K$ with a small and with a large norm. The statements of these theorems remain valid if the restriction on the behavior of the operator in a neighborhood of the point θ, taken from one theorem, is united with the restriction from another theorem on the values of the operator on elements with a large norm. This remark applies to the statements proved in section 4.4.

4.3. Auxiliary statements

4.3.1. *Mapping onto cylinders*

We assume that the cone K allows plastering. We denote by $l(x)$ a uniformly positive linear functional on K, that is, a functional such that

$$l(x) \geqslant m\|x\| \qquad (x \in K).$$

We denote by L the set of $x \in E$ such that $l(x) = \theta$.

Let z_0 be an element of K such that

$$l(z_0) = 1\ ;$$

then every element $x \in E$ allows a unique representation in the form

$$x = l(x)z_0 + x_1 \qquad (x_1 \in L). \tag{4.25}$$

We define the operator P by the equality

$$Px = x_1\ .$$

Then the representation (4.26) can be written in the form

$$x = l(x)z_0 + Px \qquad (x \in E)\,. \tag{4.26}$$

We denote by $K_{\alpha\beta}(0<\alpha<\beta)$ the collection of elements $x_1 \in K$ such that

$$\alpha \leqslant l(x) \leqslant \beta\,. \tag{4.27}$$

We denote by S the collection of elements $x_1 \in L$ such that $x_1 + z_0 \in K$ (in other words, it is necessary for the construction S to consider the intersection of the cone K with the hyperplane $l(x)=1$ and to translate the set obtained by the element z_0). We denote by T the collection of elements $x \in E$ such that $Px \in S$, that is, the collection of elements $x \in E$ of the form

$$x = x_1 + tz_0 \qquad (x_1 \in S,\ -\infty < t < \infty) \tag{4.28}$$

(in other words, T is an infinite cylinder with base S and generator directed along the vector z_0). Finally, we denote by $T_{a,b}$ the part of the cylinder T consisting of those elements for which t satisfies the inequalities $a \leqslant t \leqslant b$ in the representation (4.28).

We introduce for consideration the operator B by means of the equality

$$Bx = \frac{1}{l(x)}Px + \frac{2l(x)-(\alpha_0+\beta_0)}{\beta_0-\alpha_0}z_0\,, \tag{4.29}$$

where α_0, β_0 are fixed positive numbers where $\alpha_0 < \beta_0$. The operator B is defined on all the elements $x \in L$.

Lemma 4.2. *The operator B effects a one to one bicontinuous and mutually weakly continuous mapping of every set $K_{\alpha,\beta}$ $(0<\alpha<\beta)$ onto the corresponding set $T_{a,b}$, where*

$$a = \frac{2\alpha-(\alpha_0+\beta_0)}{\beta_0-\alpha_0}\,, \quad b = \frac{2\beta-(\alpha_0+\beta_0)}{\beta_0-\alpha_0}\,. \tag{4.30}$$

Proof. In order for the element y to belong to T, it is necessary and sufficient that the element $Py+z_0$ belong to K. By virtue of (4.29),

$$PBx + z_0 = \frac{Px}{l(x)} + z_0 = \frac{1}{l(x)}[Px + l(x)z_0] = \frac{x}{l(x)}\,.$$

Therefore $Bx \in T$ for $x \in K(x \neq \theta)$.

The fact that $BK_{\alpha,\beta} \subset T_{a,b}$ is obvious.

The continuity and weak continuity of the operator B also become evident if the formula (4.29) is replaced by the equivalent one

$$Bx = \frac{x}{l(x)} - z_0 + \frac{2l(x)-(\alpha_0+\beta_0)}{\beta_0-\alpha_0}z_0 = \frac{x}{l(x)} + \frac{2l(x)-2\beta_0}{\beta_0-\alpha_0}z_0\,. \tag{4.31}$$

It follows from formula (4.29) that the operator B transforms distinct non-zero elements of the cone K again into distinct elements. Therefore, the operator B has an inverse B^{-1}.
The explicit formula

$$B^{-1}y = \frac{(\beta_0-\alpha_0)l(y)+\alpha_0+\beta_0}{2}(Py+z_0) \tag{4.32}$$

can be given for this inverse operator.
We shall consider the operator B^{-1} on the elements $y\in T$ such that

$$l(y) > -\frac{\alpha_0+\beta_0}{\beta_0-\alpha_0}.$$

In particular, all the elements Bx where $x\in K(x\neq\theta)$ satisfy the last condition. Direct calculation shows that

$$B^{-1}Bx\equiv x \qquad (x\in E,\ l(x)>0).$$

It follows from the equality (4.32) that $B^{-1}T_{a,b}\subset K_{\alpha,\beta}$, since

$$l(B^{-1}y) = \frac{(\beta_0-\alpha_0)l(y)+\alpha_0+\beta_0}{2},$$

and the inequalities

$$\alpha\leqslant l(B^{-1}y)\leqslant\beta \qquad (y\in T_{a,b})$$

coincide with the inequality (4.27). Thus

$$BK_{\alpha,\beta}=T_{a,b}.$$

The continuity and weak continuity of the operator B^{-1} on $T_{a,b}$ is obvious. This completes the proof of the lemma.

We observe that, by virtue of Lemma 4.2, the operator B transforms the set K_{α_0,β_0} in one-to-one fashion, continuously and weakly continuously on $T_{1,1}$.

4.3.2. *Lemma about a fixed point*
We pass to the study of the positive operators A. In the sequel, it is assumed that either one of two conditions is satisfied: either the operator A is completely continuous, or the operator A is weakly continuous and the space E is weakly complete and the unit sphere in this space is weakly compact. We recall that, by assumption, the cone K allows plastering.
We shall assume that the operator A is defined on some set K_{α_0,β_0} $(0<\alpha_0<\beta_0)$.

Lemma 4.3. *Let the positive operator A satisfy the condition*

$$Ax=v_0 \qquad (x\in K,\ l(x)=\alpha_0) \tag{4.33}$$

and

$$Ax=u_0 \qquad (x\in K,\ l(x)=\beta_0)\,, \tag{4.34}$$

where v_0 and u_0 are fixed elements of K such that

$$0\leqslant l(v_0)<\alpha_0<\beta_0<l(u_0)<\infty\,. \tag{4.35}$$

Then the operator A has at least one fixed point on K_{α_0,β_0}.

Proof. It follows from the conditions of the lemma that

$$\sup_{x\in K_{\alpha_0,\beta_0}} l(Ax)<\infty\,. \tag{4.36}$$

We shall give the proof under the additional assumptions that $l(v_0)>0$ and

$$\inf_{x\in K_{\alpha_0,\beta_0}} l(Ax)>0\,. \tag{4.37}$$

This does not restrict the generality of the result since otherwise the operators

$$A_nx=Ax+\frac{u_0}{n} \qquad (x\in K_{\alpha_0,\beta_0})$$

will satisfy the additional conditions and all the conditions of the lemma being proved provided n is sufficiently large. If the statement of the lemma will be proved under the additional assumptions, then every operator A_n will have a fixed point $x_n\in K_{\alpha_0,\beta_0}$. This means that

$$Ax_n+\frac{u_0}{n}=x_n \qquad (n=1,2,\ldots)\,.$$

Without loss of generality, it can be assumed that the sequence Ax_n and, together with it, the sequence x_n converges to some element $x^*\in K_{\alpha_0,\beta_0}$ (strongly if A is completely continuous and weakly if A is weakly continuous). The element x^* will be a fixed point of the operator A.

Thus, it can also be assumed that $l(v_0)>0$, that is $v_0\neq\theta$, and that the condition (4.37) is satisfied.

By virtue of (4.36) and (4.37) positive numbers α and β can be found such that

$$\alpha\leqslant l(Ax)\leqslant\beta \qquad (x\in K_{\alpha_0,\beta_0})\,.$$

In this connection, $\alpha<\alpha_0$ and $\beta_0<\beta$.

We define on $T_{-1,1}$ the operator D by the equality

$$Dy=BAB^{-1}y-2l(BAB^{-1}y-y)z_0\,, \tag{4.38}$$

where B and B^{-1} are the operators (4.29) and (4.32). By virtue of Lemma 4.2, the operator D is defined and has the property of complete or weak continuity if the operator A is completely or weakly continuous respectively.

If $Ax^* = x^*$, then the element $y^* = Bx^*$ will be a fixed point of the operator D since

$$Dy^* = BAx^* - 2l(BAx^* - Bx^*)z_0 = Bx^* = y^* .$$

However, if $Dy^* = y^*$ then the element $x^* = B^{-1}y^*$ satisfies the equality

$$BAx^* - 2l(BAx^* - Bx^*)z_0 = Bx^* ,$$

that is, the equality

$$BAx^* - Bx^* = 2l(BAx^* - Bx^*)z_0 .$$

It follows from this equality that

$$l(BAx^* - Bx^*) = 0 ,$$

since $l(z_0) = 1$. Therefore, $BAx^* = Bx^*$ and $Ax^* = x^*$.

Thus, the question of the existence (and even of the number) of fixed points for the operator A on K_{α_0,β_0} is entirely equivalent to the question of the existence of fixed points for the operator D on $T_{-1,+1}$.

We enumerate some properties of the operator D.

Let $l(y) = -1$. Then $l(B^{-1}y) = \alpha_0$ and, by virtue of (4.33), $AB^{-1}y = v_0$. This means that

$$Dy = Bv_0 - 2l(Bv_0 - y)z_0 = Bv_0 - [2 + 2l(Bv_0)]z_0 = h_0 \qquad (y \in T_{-1,1},\ l(y) = -1) .$$

In this connection, by virtue of (4.29),

$$l(h_0) = -2 - 2l(Bv_0) = -1 + \frac{2[\alpha_0 - l(v_0)]}{\beta_0 - \alpha_0}$$

and, by virtue of (4.35),

$$l(h_0) > -1 .$$

Analogously

$$Dy = Bu_0 - [2 + 2l(Bu_0)]z_0 = g_0 \qquad (y \in T_{-1,1},\ l(y) = 1) ,$$

where

$$l(g_0) < 1 .$$

We extend the definition of the operator D to all of $T_{a,b}$ where the numbers a and b are defined by the equalities (4.30), setting

$$Dy = \begin{cases} h_0, & \text{if } y \in T_{a,b}, \quad a \leqslant l(y) \leqslant -1, \\ g_0, & \text{if } y \in T_{a,b}, \quad 1 \leqslant l(y) \leqslant b. \end{cases}$$

The extended operator D will, together with the operator D, be completely continuous or weakly continuous.

The extended operator D transforms $T_{a,b}$ into itself. By virtue of Schauder's or Tikhonov's principle, the extended operator D has at least one fixed point. By virtue of the inequalities $l(h_0) > -1$ and $l(g_0) < 1$ this fixed point cannot lie outside of $T_{-1,1}$.

Thus, the operator D on $T_{-1,1}$ has a fixed point.

This completes the proof of the lemma.

4.3.3. *Case of a finite-dimensional space*

We shall use Lemma 4.3 below, in particular, in the study of operators which act in finite-dimensional spaces. In finite-dimensional spaces, the weak and complete continuity of an operator coincides with usual continuity. We also recall that all cones in a finite-dimensional space allow plastering since very strictly positive functional $l(x)[l(x) > 0$ for $x \in K$ and $x \neq \theta]$ is uniformly positive on a finite-dimensional cone.

We denote by $F_{r,R}$ the set of elements $x \in K$ such that $r \leqslant \|x\| \leqslant R$.

We mention one modification of Lemma 4.3.

Lemma 4.4. *Let K be a cone in a finite-dimensional space E. Let the operator A, positive and continuous on $F_{r,R}$, satisfy the conditions*

$$Ax = v_0 \qquad (x \in K, \ \|x\| = r) \tag{4.39}$$

and

$$Ax = u_0 \qquad (x \in K, \ \|x\| = R), \tag{4.40}$$

where

$$\|v_0\| < r < R < \|u_0\|. \tag{4.41}$$

Then the operator A has at least one fixed point on $F_{r,R}$.

Proof. Let $l(x)$ be a uniformly positive linear functional on K. We set

$$A_1 y = \frac{\left\| A\left[\frac{l(y)}{\|y\|} y\right] \right\|}{l\left\{A\left[\frac{l(y)}{\|y\|} y\right]\right\}} A\left[\frac{l(y)}{\|y\|}\right] y;$$

this operator is defined on $K_{r,R}$ and satisfies the conditions of the Lemma 4.3. Therefore, a point $y^* \in K_{r,R}$ can be found such that

$$\frac{\left\|A\left[\frac{l(y^*)}{\|y^*\|}y^*\right]\right\|}{l\left\{A\left[\frac{l(y^*)}{\|y^*\|}y^*\right]\right\}} A\left[\frac{l(y^*)}{\|y^*\|}y^*\right] = y^*.$$

We introduce the notation

$$\frac{l(y^*)}{\|y^*\|}y^* = x^*.$$

Obviously, $x^* \in F_{r,R}$. It follows from the equality

$$y^* = \frac{\|Ax^*\|}{l(Ax^*)}Ax^*$$

that

$$x^* = \frac{l\left[\frac{\|Ax^*\|}{l(Ax^*)}Ax^*\right]}{\left\|\frac{\|Ax^*\|}{l(Ax^*)}Ax^*\right\|} \cdot \frac{\|Ax^*\|}{l(Ax^*)}Ax^* = Ax^*.$$

Thus, x^* is a fixed point of the operator A.
This completes the proof of the lemma.

4.3.4. *Schauder operator of projection onto a finite-dimensional subspace*
Let F be a compact set in a Banach space E. Let α be a given positive number. We denote by $y_1, y_2, \ldots, y_n$ a finite $\alpha/2$-net of the set F; this means that for an arbitrary $y \in F$ at least one element y_{i_0} can be found such that

$$\|y - y_{i_0}\| < \frac{\alpha}{2}.$$

We define on F the operator P by the equality

$$Py = \frac{\sum_{i=1}^{n} \mu_i(y) y_i}{\sum_{i=1}^{n} \mu_i(y)}, \tag{4.42}$$

where

$$\mu_i(y) = \begin{cases} \frac{\alpha}{2} - \|y - y_i\|, & \text{if } \|y - y_i\| \leqslant \frac{\alpha}{2}, \\ 0, & \text{if } \|y - y_i\| \geqslant \frac{\alpha}{2}. \end{cases}$$

The continuity of the operator P is obvious. The operator P transforms every element $y \in F$ into an element of the convex hull of those points

$y_1, y_2, \ldots, y_n$ of the $\alpha/2$-net of the set F to which the distance from y is less than $\alpha/2$. Therefore,

$$\|Py-y\| < \frac{\alpha}{2} \qquad (y \in F). \tag{4.43}$$

It is not difficult to see that the operator P projects F onto a finite-dimensional subspace, the linear hull of the points $y_1, y_2, \ldots, y_n$.

4.3.5. *Lemma about a fixed point of a completely continuous operator*

Lemma 4.5. *Let a completely continuous operator A be positive on the set of elements $x \in K$ such that $r \leqslant \|x\| \leqslant R$. Let*

$$Ax = \begin{cases} v_0, & \text{if } x \in K, \ \|x\| = r, \\ u_0, & \text{if } x \in K, \ \|x\| = R, \end{cases}$$

where $\|v_0\| < r < R < \|u_0\|$.
Then the operator A has at least one fixed point.

Proof. We assume that the operator A does not have fixed points. Then

$$\inf_{x \in K, \, r \leqslant \|x\| \leqslant R} \|x - Ax\| > 0 .$$

If the last inequality did not occur, then we could find a sequence $x_n \in K$, $r \leqslant \|x_n\| \leqslant R$ such that

$$\lim_{n \to \infty} \|x_n - Ax_n\| = 0 .$$

Since A is completely continuous, then, without loss of generality, we can assume that the sequence Ax_n converges in norm. The elements

$$x_n = (x_n - Ax_n) + Ax_n \qquad (n = 1, 2, \ldots)$$

converge in norm to the limit x^*. Then $x^* = Ax^*$, which contradicts the assumption.

Thus, a number $\alpha > 0$ can be found such that

$$\|x - Ax\| > \alpha \qquad (x \in K, \ r \leqslant \|x\| \leqslant R) .$$

In this connection, we can assume, without loss of generality, that

$$\alpha < r - \|v_0\| , \quad \alpha < \|u_0\| - R .$$

We consider the operator (4.42) constructed on some $\alpha/2$-net of the set F of the elements Ax where $x \in K$ and $r \leqslant \|x\| \leqslant R$. We denote by E_n a finite-dimensional subspace onto which the operator P projects F; let $K_n = K \cap E_n$ (K_n is a non-degenerate cone since all the points of the $\alpha/2$-net of the set F belong to the subspace E_n as well as to the cone K).

The operator $A_1=PA$, which is considered on K_n (more exactly, on the elements $x\in K_n$ such that $r\leqslant\|x\|\leqslant R$), satisfies the conditions of the Lemma 4.4 and, therefore, has a fixed point x^*:

$$PAx^*=x^*.$$

On the other hand, by virtue of (4.43),

$$\|PAx^*-x^*\|\geqslant\|Ax^*-x^*\|-\|Ax^*-PAx^*\|>\frac{\alpha}{2}.$$

We have arrived at a contradiction.
This completes the proof of the theorem.

4.4. Fixed points of operators which expand a cone

4.4.1. *Expansion of a cone*
We call the positive operator $A(A\theta=\theta)$ an *expansion* of the cone K if numbers r, $R>0$ can be found such that for all $\varepsilon>0$

$$Ax\geqq(1+\varepsilon)x \qquad (x\in K,\ \|x\|\leqslant r,\ x\neq\theta) \tag{4.44}$$

and

$$Ax\leqq x \qquad (x\in K,\ \|x\|\geqslant R). \tag{4.45}$$

Theorem 4.14. *Let the positive completely continuous operator A be an expansion of the cone.*
Then the operator A has at least one non-zero fixed point on K.

Proof. We shall say that $R>1$. We set

$$\tilde{A}x=\begin{cases}\dfrac{2\|x\|-r}{r}A\left[\dfrac{r}{\|x\|}x\right], & \text{if } \dfrac{r}{2}\leqslant\|x\|\leqslant r,\\ Ax, & \text{if } r\leqslant\|x\|\leqslant R,\\ \dfrac{\|x\|}{R}A\left[\dfrac{R}{\|x\|}x\right]+(\|x\|-R)h_0, & \text{if } R\leqslant\|x\|\leqslant R^2,\\ \dfrac{\|x\|}{R}\cdot\dfrac{R^3-\|x\|^2}{R^3-R^2}A\left[\dfrac{R}{\|x\|}x\right]+(R^2-R)h_0, & \text{if } R^2\leqslant\|x\|\leqslant R^3,\end{cases} \tag{4.46}$$

where h_0 is some fixed non-zero element of K satisfying the inequality

$$\|h_0\|>\frac{1}{R^2-R}\left[R^3+R^2\sup_{y\in K,\ \|y\|\leqslant R}\|Ay\|\right].$$

The operator $\tilde{A}$ is evidently completely continuous and satisfies the conditions of Lemma 4.5. Let x^* be a fixed point of the operator $\tilde{A}$. The proof

of the theorem will be complete if we show that

$$r \leqslant \|x^*\| \leqslant R .$$

It is obvious that $\|x^*\| > r/2$. We assume that $\|x^*\| < r$. Then

$$x^* = \frac{2\|x^*\| - r}{r} A\left(\frac{r}{\|x^*\|} x^*\right).$$

This means that the element

$$y^* = \frac{r}{\|x^*\|} x^*$$

satisfies the equality

$$Ay^* = \left(1 + \frac{r - \|x^*\|}{2\|x^*\| - r}\right) y^* ,$$

which contradicts (4.44).
We assume that $R < \|x^*\| \leqslant R^2$. Then

$$x^* = \frac{\|x^*\|}{R} A\left(\frac{R}{\|x^*\|} x^*\right) + (\|x^*\| - R) h_0 ,$$

from which follows the inequality

$$A\left(\frac{R}{\|x^*\|} x^*\right) \leqslant \frac{R}{\|x^*\|} x^* ,$$

which contradicts the condition (4.45).
Finally, we assume that $R^2 < \|x^*\| \leqslant R^3$. Then

$$x^* = \frac{\|x^*\|}{R} \frac{R^3 - \|x^*\|}{R^3 - R^2} A\left(\frac{R}{\|x^*\|} x^*\right) + (R^2 - R) h_0 ,$$

from which

$$\|h_0\| \leqslant \frac{1}{R^2 - R}\left[\|x^*\| + \frac{\|x^*\|}{R} \frac{R^3 - \|x^*\|}{R^3 - R^2} \left\|A\left(\frac{R}{\|x^*\|} x^*\right)\right\|\right] \leqslant$$

$$\leqslant \frac{1}{R^2 - R}\left[R^3 + R^2 \left\|A\left(\frac{R}{\|x^*\|} x^*\right)\right\|\right],$$

which contradicts (4.47).
Thus, $r \leqslant \|x^*\| \leqslant R$. Therefore $Ax^* = \tilde{A}x^* = x^*$.
This completes the proof of the theorem.

Theorem 4.15. *Let the space E be weakly complete, the unit sphere weakly compact, and the cone K allow plastering. Let the weakly continuous A be an expansion of the cone.*
Then the operator A has at least one fixed point on K.

We leave the proof of the theorem to the reader. We observe only that instead of the operator (4.46), we consider the operator

$$\tilde{A}x = \begin{cases} \dfrac{2l(x)-r}{r} A\left[\dfrac{r}{l(x)}x\right], & \text{if } \dfrac{r}{2} \leqslant l(x) \leqslant r, \\ Ax, & \text{if } r \leqslant l(x) \leqslant R, \\ \dfrac{l(x)}{R} A\left[\dfrac{R}{l(x)}x\right] + [l(x)-R]h_0, & \text{if } R \leqslant l(x) \leqslant R^2, \\ \dfrac{l(x)}{R}\dfrac{R^3-l(x)}{R^3-R^2} A\left[\dfrac{R}{l(x)}x\right] + (R^2-R)h_0, & \text{if } R^2 \leqslant l(x) \leqslant R^3, \end{cases} \tag{4.47}$$

where $l(x)$ is some linear uniformly positive functional such that for all $\varepsilon > 0$

$$Ax \overline{\geqslant} (1+\varepsilon)x \qquad (x \in K,\ l(x) \leqslant r) \tag{4.48}$$

and

$$Ax \overline{\leqslant} x \qquad (x \in K,\ l(x) \geqslant R). \tag{4.49}$$

4.4.2. *Use of majorants and minorants*

The examination of conditions (4.44) and (4.45) or (4.48) and (4.49) often is simplified if the majorants A^+ and minorants A^- of the operator A are used. It is not difficult to see that for the fulfillment of condition (4.44) or, analogously, condition (4.48) the fulfillment of the relations

$$A^+x \overline{\geqslant} (1+\varepsilon)x \qquad (x \in K,\ \|x\| \leqslant r) \tag{4.50}$$

suffices. For the fulfillment of condition (4.45) (analogously (4.49)) it suffices that some minorant satisfy the relations

$$A^-x \overline{\leqslant} x \qquad (x \in K,\ \|x\| \geqslant r). \tag{4.51}$$

It is easier to choose the operators (for example, linear) as minorants and majorants after which the examination of conditions (4.50) and (4.51) becomes simple.

We observe that conditions (4.44) and (4.45) can be replaced, as this is clear from the proofs, by less restrictive and simply formulated conditions: instead of (4.44) it can be assumed that $Ax \neq (1+\varepsilon)x$ and instead of (4.45) it can be assumed that for some $h_0 \in K$ for $\alpha > 0$ and $x \in K$ ($\|x\| \leqslant r$) the inequalities $x \neq Ax + \alpha h_0$ are satisfied. However, the transition to new conditions is not convenient since in the application of majorants and minorants it is more convenient to verify the rough relations (4.44) and (4.45). Everything stated concerns conditions (4.50) and (4.51).

4.4.3. *Use of derivatives*

The conditions of the type (4.44) and (4.45), as is already known to us, are easily verified for linear operators (see chapter 2, section 2.5, subsection 2.5.4) and for operators which are near to linear (see chapter 3, sections 3.3 and 3.4). We show the theorem in which conditions for the existence of positive solutions are formulated in terms of the properties of the operators $A'(\theta)$ and $A'(\infty)$.

We shall assume that the positive operator A which is considered has a strong Fréchet derivative $A'(\theta)$ with respect to a cone and a strong asymptotic derivative $A'(\infty)$ with respect to a cone. We shall assume that the operator A is either completely continuous or weakly continuous where in the last case, as usual, we shall consider the space E weakly complete and weakly compact, and the cone K plastered.

Theorem 4.16. *Let the operator $A'(\theta)$ not have positive characteristic values, exceeding or equal to* 1. *Let the operator $A'(\infty)$ have a characteristic vector $g_0 \in K$:*

$$A'(\infty)g_0 = \lambda_\infty g_0 , \tag{4.52}$$

where $\lambda_\infty > 1$ and $A'(\infty)$ does not have positive characteristic values equal to 1.

Then the operator A has at least one fixed non-zero point on K.

Proof. The line of reasoning is analogous to that in the case of a completely continuous as well as in the case of a weakly continuous, operator A. We restrict ourselves to the consideration of a weakly continuous operator.

We consider a sequence of operators A_n: such that

$$A_n x = \begin{cases} [2nl(x)-1]\, A\left[\dfrac{1}{nl(x)}\, x\right], & \text{if } \dfrac{1}{2n} \leqslant l(x) \leqslant \dfrac{1}{n}, \\ Ax, & \text{if } \dfrac{1}{n} \leqslant l(x) \leqslant n, \\ \dfrac{l(x)}{n} A\left[\dfrac{n}{l(x)}\, x\right] + [l(x)-n]^2 g_0, & \text{if } n \leqslant l(x) \leqslant n^2, \\ \dfrac{l(x)}{n} \dfrac{n^3 - l(x)}{n^3 - n^2} A\left[\dfrac{n}{l(x)}\, x\right] - (n^2-n)^2 g_0, & \text{if } n^2 \leqslant l(x) \leqslant n^3, \end{cases} \tag{4.53}$$

where $l(x)$ is a linear uniformly positive functional: $l(x) \geqslant a\|x\|$ for $x \in K$. By virtue of the Lemma 4.3, each of these operators has a fixed point x_n.

To complete the proof, it suffices to show that the inequalities

$$\frac{1}{n} \leqslant l(x_n) \leqslant n$$

are satisfied for large n. We shall show that the non-fulfillment of these inequalities leads to a contradiction.

The arguments developed below are analogous to the proof of Theorem 4.11.

We assume that for some sequence of indices n_i the inequalities

$$\frac{1}{2n_i} \leqslant l(x_{n_i}) \leqslant \frac{1}{n_i} \qquad (i=1, 2, \ldots)$$

are satisfied. Then

$$x_{n_i} = [2n_i l(x_{n_i}) - 1]\, A\left[\frac{x_{n_i}}{n_i l(x_{n_i})}\right],$$

that is

$$y_i = \alpha_i A y_i \qquad (i=1, 2, \ldots), \tag{4.54}$$

where

$$y_i = \frac{x_{n_i}}{n_i l(x_{n_i})}, \quad 0 < \alpha_i = \frac{2n_i l(x_{n_i}) - 1}{n_i l(x_{n_i})} \leqslant 1.$$

Without loss of generality, it can be assumed that the numbers α_i converge to some limit $\alpha_0 \in [0, 1]$ and the elements $y_i/l(y_i)$ converge weakly to some element v_0, which is different from θ since $l(v_0) = 1$. Then the elements $A'(\theta)[y_i/l(y_i)]$ converge weakly to $A'(\theta)v_0$. We rewrite the equalities (4.54) in the form

$$\frac{y_i}{l(y_i)} = \alpha_i \frac{Ay_i - A'(\theta)y_i}{l(y_i)} + (\alpha_i - \alpha_0) A'(\theta) \frac{y_i}{l(y_i)} + \alpha_0 A'(\theta) \frac{y_i}{l(y_i)}$$

and pass to the weak limit as $i \to 0$. On the right side, the first two terms approach zero in the norm. Thus

$$v_0 = \alpha_0 A'(\theta) v_0 .$$

The last equality contradicts the assumption that $A'(\theta)$ does not have positive characteristic values greater or equal to 1.

We assume that for some sequence of indices n_i the inequalities

$$n_i \leqslant l(x_{n_i}) \leqslant n_i^2 \qquad (i=1, 2, \ldots)$$

are satisfied. Then

$$\frac{n_i x_{n_i}}{l(x_{n_i})} = A\left[\frac{n_i x_{n_i}}{l(x_{n_i})}\right] + \frac{[l(x_{n_i}) - n_i]^2}{l(x_{n_i})} n_i g_0 ,$$

that is

$$y_i = Ay_i + \beta_i l(y_i) g_0 \qquad (i=1, 2, \ldots), \tag{4.55}$$

where

$$y_i = \frac{n_i x_{n_i}}{l(x_{n_i})}, \quad 0 < \beta_i = \frac{l(y_i) - l(Ay_i)}{l(y_i)\, l(g_0)} \leqslant \frac{1}{l(g_0)}.$$

Without loss of generality, it can be assumed that the sequence $y_i/l(y_i)$ converges weakly to some element u_0, which is different from zero; in this connection, the sequence $A'(\infty)[y_i/l(y_i)]$ converges weakly to $A'(\infty)u_0$. The sequence β_i is also assumed to be convergent to some number β_0. We rewrite the equalities (4.55) in the form

$$\frac{y_i}{l(y_i)} = \frac{Ay_i - A'(\infty)y_i}{l(y_i)} + A'(\infty)\frac{y_i}{l(y_i)} + \beta_i g_0$$

and pass to the weak limit as $i \to \infty$. The limit equality has the form

$$u_0 = A'(\infty)u_0 + \beta_0 g_0 . \tag{4.56}$$

Here β_0 is positive since otherwise 1 is a characteristic value of the operator $A'(\infty)$. Let $u_0 \geqslant t_0 g_0$ and $u \not\geqslant t g_0$ for $t > t_0$. By virtue of (4.56)

$$u_0 \geqslant A'(\infty)(t_0 g_0) + \beta_0 g_0 \not\geqslant (\lambda_\infty t_0 + \beta_0) g_0 ,$$

which contradicts the maximality of t_0.

Finally, we assume that for some sequence of indices n_i the inequalities

$$n_i^2 < l(x_{n_i}) \leqslant n_i^3 \qquad (i=1, 2, \ldots)$$

are satisfied. Then

$$\frac{n_i x_{n_i}}{l(x_{n_i})} = \frac{n_i^3 - l(x_{n_i})}{n_i^3 - n_i^2} A\left[\frac{n_i}{l(x_{n_i})} x_{n_i}\right] + \frac{n_i(n_i^2 - n_i)^2}{l(x_{n_i})} g_0 \tag{4.57}$$

or

$$y_i = \gamma_i A y_i + \delta_i l(y_i) g_0 \qquad (i=1, 2, \ldots),$$

where

$$y_i = \frac{n_i x_{n_i}}{l(x_{n_i})}, \quad 0 \leqslant \gamma_i = \frac{n_i^3 - l(x_{n_i})}{n_i^3 - n_i^2} < 1 ,$$

and

$$\delta_i = \frac{(n_i^2 - n_i)^2}{l(x_{n_i})} \geqslant \frac{(n_i^2 - n_i)^2}{\|l\|\, \|x_{n_i}\|} \geqslant \frac{(n_i^2 - n_i)^2}{\|l\| n_i^3}. \tag{4.58}$$

It follows from (4.57) that

$$\delta_i \leqslant \frac{1}{\|g_0\|}\left\{\frac{\|y_i\|}{l(y_i)} + \gamma_i \frac{\|Ay_i - A'(\infty)y_i\|}{l(y_i)} + \gamma_i \left\| A'(\infty)\frac{y_i}{l(y_i)} \right\|\right\} \leqslant$$

$$\leqslant \frac{1}{\|g_0\|}\left\{\frac{1}{a} + \frac{\|Ay_i - A'(\infty)y_i\|}{a\|y_i\|} + \frac{1}{a}\|A'(\infty)\|\right\},$$

that is

$$\sup_{i=1,2,\ldots} \delta_i < \infty ,$$

which contradicts (4.58).
This completes the proof of the theorem.

4.4.4. *Existence of many solutions*
In the theorems mentioned above about operators which are compressions or expansions of a cone, the restrictions were imposed on the nature of the values Ax on elements $x \in K$ with a sufficiently small and a sufficiently large norm. It is clear from the proofs that these restrictions could be imposed on values of the operators on elements with a fixed norm.
For example, let the positive completely continuous operator A satisfy the conditions

$$Ax \overline{\leqslant} x \qquad (x \in K,\ \|x\| = R) \tag{4.59}$$

and

$$Ax \overline{\geqslant} (1+\varepsilon)x \qquad (x \in K,\ \|x\| = R^*,\ \varepsilon > 0) . \tag{4.60}$$

Then for $R < R^*$ the existence of a fixed point x^* for the operator A such that

$$R < \|x^*\| < R^*$$

will follow from the theorem about operators which compress a cone. For $R^* < R$ this statement will follow from the theorem about operators which expand a cone.
It is not difficult to see that the relations (4.59) and (4.60) cannot simultaneously be satisfied for $R = R^*$.
If there are two alternating sequences $R_n, R_n^* \to \infty$ having the property that

$$Ax \overline{\leqslant} x \qquad (x \in K,\ \|x\| = R_n,\ n = 1, 2, \ldots)$$

and

$$Ax \overline{\geqslant} (1+\varepsilon)x \qquad (x \in K,\ \|x\| = R_n^*,\ \varepsilon > 0,\ n = 1, 2, \ldots) ,$$

then the operator A will have a denumerable number of distinct fixed points.
We do not stop to formulate the corresponding statements for weakly continuous operators since these are obvious.

Chapter 5

CONTINUOUS BRANCHES OF POSITIVE SOLUTIONS

In this chapter the question of positive solutions of operator equations depending on a parameter is considered. The simplest existence theorems follow from the results of the preceding chapter.
Geometric characteristics of the collection of all solutions corresponding to distinct values of a parameter are studied.
In some cases we do not succeed in exhibiting the values of the parameter for which the equation studied has a positive solution; criteria for the existence of solutions for these cases are indicated for some, a priori unknown, values of the parameter. Corresponding theorems are based on topological considerations; the proofs are developed in a separate section on which the constructions of the following chapters are not based.

5.1. Positive solutions of equations with a parameter

5.5.1. *Description of the equations studied*
In the space E with the cone K, let the non-linear operator $A(x;\mu)$ be defined depending on the parameter μ. It is assumed that μ is a numerical parameter (many statements remain valid for non-numerical parameters). For every fixed μ the operator $A(x;\mu)$ is defined and positive on the entire cone K.
We consider the equation

$$x = A(x;\mu)\,. \tag{5.1}$$

We shall denote its positive solution by $x(\mu)$.
We assume that $x(\mu)$ not only exists for the values μ of some interval but that it is unique.
It is natural to expect that in the case of sufficiently smooth operators $A(x;\mu)$ the vector-function $x(\mu)$ is continuous. In fact, such a continuous dependence occurs. If there is no uniqueness, then the collection $\mathfrak{N}$ of all solutions is not a continuous curve in the space E; however, many properties of a continuous curve are retained.

Throughout the entire chapter we basically restrict ourselves to the consideration of equations with completely continuous operators. In this connection, the operator $A(x;\mu)$ depending on the parameter μ is called *completely continuous* if it is continuous with respect to the collection of variables and if the set of its values is compact when x ranges over an arbitrary bounded set and μ an arbitrary finite interval.

5.1.2. *Existence of solutions*

We assume that the completely continuous positive operator A has a strong asymptotic derivative $A'(\infty;\mu)$ for every $\mu\in\mathfrak{M}$. We assume that the operators $A'(\infty;\mu)(\mu\in\mathfrak{M})$ do not have positive characteristic values which are less than or equal to 1. It then follows from the theorems of the preceding chapter that the equation (5.1) has at least one solution in the cone K for all $\mu\in\mathfrak{M}$.

The investigation is more complicated if $A(\theta;\mu)=\theta$ and the question is about non-zero solutions.

We further assume that the operator $A(x;\mu)$ has strong Fréchet derivatives $A'(\theta;\mu)$ and an asymptotic derivative $A'(\infty;\mu)$ of the special form

$$A'(\theta;\mu)=\mu A'(\theta)\,,\quad A'(\infty;\mu)=\mu A'(\infty)\,. \tag{5.2}$$

Theorem 5.1. *Let $A(\theta;\mu)\equiv\theta$. Let every operator $A'(\theta)$ and $A'(\infty)$ have in the cone K a unique characteristic vector*

$$A'(\theta)h_0=\lambda_0 h_0\,,\quad A'(\infty)g_0=\lambda_\infty g_0\,. \tag{5.3}$$

Then the equation (5.1) *for*

$$\mu\in\left(\frac{1}{\lambda_0},\frac{1}{\lambda_\infty}\right) \tag{5.4}$$

has at least one non-zero solution in K.

The proof consists of the direct verification of the conditions of the theorems of the preceding chapter.

We remark that Theorem 5.1 remains valid if one of the operators (5.2) does not have positive characteristic values. In this case, one of the limits of the interval (5.4) must be assumed to be equal to $+\infty$.

The assumption about the differentiability of the operator $A(x;\mu)$ can be replaced by the assumption of the existence of suitable majorants and minorants. From Theorems 4.12 and 4.14 and from the theorems of disjoint inequalities for the linear operators (chapter 2, section 2.5, subsection 2.5.4) follows, for example, the next statement:

Theorem 5.2. *Let $A(\theta;\mu)\equiv\theta$. Let the positive completely continuous operator $A(x;\mu)$ have a linear minorant for sufficiently small numbers x and a linear majorant for sufficiently large numbers x on K, that is, let the following conditions be satisfied*:

$$A(x;\mu)\geqslant\mu A_{-}(\theta)x \qquad (x\in K,\ \|x\|\leqslant r(\mu)) \tag{5.5}$$

and

$$A(x;\mu)\leqslant\mu A_{+}(\infty)x \qquad (x\in K,\ \|x\|\geqslant R(\mu)), \tag{5.6}$$

where $A_{-}(\theta)$ and $A_{+}(\infty)$ are linear positive operators. Let non-zero elements u_0, $v_0\in K$ exist such that the operator $A_{-}(\theta)$ is u_0-bounded below and

$$A_{-}(\theta)u_0\geqslant\lambda_0 u_0, \tag{5.7}$$

and the operator $A_{+}(\infty)$ is v_0-bounded above and

$$A_{+}(\infty)v_0\leqslant\lambda_\infty v_0. \tag{5.8}$$

Then for $\lambda_0<\lambda_\infty$ and

$$\mu\in\left(\frac{1}{\lambda_0},\ \frac{1}{\lambda_\infty}\right) \tag{5.4}$$

the equation (5.1) *has at least one non-zero solution in K.*

Theorem 5.3. *Let $A(\theta;\mu)\equiv\theta$. Let the completely continuous operator $A(x;\mu)$ satisfy the conditions*

$$A(x;\mu)\leqslant\mu A_{+}(\theta)x \qquad (x\in K,\ \|x\|\leqslant r(\mu)) \tag{5.9}$$

and

$$A(x;\mu)\geqslant\mu A_{-}(\infty)x \qquad (x\in K,\ \|x\|\geqslant R(\mu)), \tag{5.10}$$

where $A_{+}(\theta)$ and $A_{-}(\infty)$ are linear operators, the first of which is u_0-bounded above and the second is v_0-bounded below where

$$A_{+}(\theta)u_0\leqslant\lambda_0 u_0 \tag{5.11}$$

and

$$A_{-}(\infty)v_0\geqslant\lambda_\infty v_0. \tag{5.12}$$

Then for $\lambda_0<\lambda_\infty$ and

$$\mu\in\left(\frac{1}{\lambda_0},\ \frac{1}{\lambda_\infty}\right) \tag{5.4}$$

the equation (5.1) *has at least one non-zero solution in K.*

The membership of μ in the interval (5.4) is, of course, only a sufficient condition for the existence of a positive solution for the equation (5.1). We note that the magnitude of the interval (5.4) depends upon which

linear minorants and majorants are constructed and on how successfully the elements u_0 and v_0 are chosen.

5.1.3. *Continuous branch of solutions*

We first make some simple remarks.

Let the operator $A(x;\mu)$ be completely continuous. We assume that $x(\mu_n)$ is a bounded sequence of solutions of the equation (5.1) corresponding to the convergent sequence of values of the parameter μ_n, $\mu_n \to \mu_0$. Without loss of generality, we assume that the sequence $A[x(\mu_n);\mu_n]$ converges in norm to some element x_0 (otherwise, we would take a subsequence). It follows from the equalities

$$x(\mu_n)=A[x(\mu_n);\mu_n] \qquad (n=1, 2, \ldots)$$

that $x(\mu_n)$ converges to x_0 and

$$x_0=A(x_0, \mu_0)\,.$$

Thus, the equation (5.1) can be solved for $\mu=\mu_0$.

The set of all solutions of the equation (5.1) which corresponds to each fixed value $\mu\in\mathfrak{M}$ is closed. We leave the proof to be shown by the reader.

We assume that for every $\mu\in\mathfrak{M}$ the equation (5.1) has a unique solution in some sphere. Then this solution $x(\mu)$ is a continuous function of μ, that is, forms a continuous curve $\mathfrak{N}$ in the space E. The value μ can be considered as a functional defined on the points $x\in\mathfrak{N}$; this functional is continuous. Both statements follow from the arguments mentioned above.

We now assume that nothing is known concerning the uniqueness of the solutions of the equation (5.1).

We shall say that the set $\mathfrak{N}\subset E$ forms a *continuous branch connecting the bounded closed set $F\subset E$ with the set $F_1\subset E$* if the intersection of the set $\mathfrak{N}$ with the boundary Γ of an arbitrary bounded domain $\mathfrak{E}$ containing F is not empty and has common points with F_1.

We denote by X_μ the set of all solutions $x(\mu)\in K$ of the equation (5.1) for a given fixed value of μ. Every X_μ is closed if the operator $A(x;\mu)$ is continuous with respect to x. If $A(x;\mu)$ is completely continuous, then every set X_μ is locally compact.

Lemma 5.1. *Let the conditions of one of the Theorems* 5.1, 5.2, or 5.3 *be satisfied and*

$$\mu\in\left(\frac{1}{\lambda_0}, \frac{1}{\lambda_\infty}\right). \tag{5.4}$$

Then

$$a(\mu) \leqslant \|x(\mu)\| \leqslant b(\mu) \qquad (x(\mu) \in X_\mu), \tag{5.13}$$

where $a(\mu), b(\mu) > 0$.

Proof. We assume that the conditions of Theorem 5.1 are satisfied and that the statement of the lemma is not true. Then a sequence $x_n \in X_\mu$ can be found such that either $\|x_n\| \to 0$ or $\|x_n\| \to \infty$. Let $\|x_n\| \to 0$. Then we write the equalities

$$x_n = A(x_n; \mu)$$

in the form of

$$\frac{x_n}{\|x_n\|} = \frac{A(x_n; \mu) - \mu A'(\theta) x_n}{\|x_n\|} + \mu A'(\theta) \frac{x_n}{\|x_n\|}. \tag{5.14}$$

Without loss of generality, we can assume that the sequence $\mu A'(\theta)(x_n)/\|x_n\|$ converges in norm to some element u_0. It follows from the equalities (5.14) that the sequence $x_n/\|x_n\|$ also converges to u_0 where $\|u_0\| = 1$ and

$$u_0 = \mu A'(\theta) u_0 .$$

This means that $u_0 = h_0$ and $\mu = 1/\lambda_0$, which contradicts (5.4). If it is assumed that $\|x_n\| \to \infty$, then the arguments are analogous (it is necessary only to replace $A'(\theta)$ by $A'(\infty)$ in formulas (5.14)).

If the conditions of Theorem 5.2 or Theorem 5.3 are satisfied, then the lemma follows from Theorems 2.17 and 2.18.

Let, for example, the conditions of Theorem 5.2 be satisfied. We assume that $x_0 \in X_\mu$ and $\|x_0\| < r(\mu)$. It then follows from (5.9) that

$$\mu A_-(\theta) x_0 \leqslant A(x_0; \mu) = x_0 ,$$

and by virtue of Theorem 2.17

$$\frac{1}{\mu} > \lambda_0 ,$$

which contradicts (5.4).

This completes the proof of the lemma.

Theorem 5.4. *Let the completely continuous operator* $A(x; \mu)$ *satisfy the conditions of Theorem* 5.1 *and*

$$[\mu_1, \mu_2] \subset \left(\frac{1}{\lambda_0}, \frac{1}{\lambda_\infty}\right), \tag{5.15}$$

where

$$\lim_{x\in K,\,\|x\|\to 0}\ \sup_{\mu\in[\mu_1,\mu_2]}\ \frac{\|A(x;\mu)-\mu A'(\theta)x\|}{\|x\|}=0$$

and

$$\lim_{x\in K,\,\|x\|\to\infty}\ \sup_{\mu\in[\mu_1,\mu_2]}\ \frac{\|A(x;\mu)-\mu A'(\infty)x\|}{\|x\|}=0\,.$$

Then the set $\mathfrak{N}$ *of all solutions of the equation* (5.1) *which correspond to the values* $\mu\in[\mu_1,\mu_2]$ *forms a continuous branch connecting the sets* X_{μ_1} *and* X_{μ_2}.

The proof of this theorem will be given in a following section.
Let the conditions of Theorem 5.4 be satisfied and let

$$\mu_0\in(\mu_1,\mu_2)\,. \tag{5.16}$$

We denote by $\mathfrak{N}_1$ the set of solutions of the equation (5.1) which correspond to the values $\mu\in[\mu_1,\mu_0]$ and by $\mathfrak{N}_2$ the set of solutions which correspond to the values $\mu\in[\mu_0,\mu_2]$. One further important characteristic of the set $\mathfrak{N}$ follows from Theorem 5.4: *the intersection of the closures of the sets* $\mathfrak{N}_1$ *and* $\mathfrak{N}_2$ *is non-empty*:

$$\overline{\mathfrak{N}}_1\cap\overline{\mathfrak{N}}_2\neq 0\,. \tag{5.17}$$

The proof of this fact is easiest to carry out from the contrary. We assume that $\overline{\mathfrak{N}}_1\cap\overline{\mathfrak{N}}_2$ is empty; then the compact sets $\overline{\mathfrak{N}}_1$ and $\mathfrak{N}\backslash\overline{\mathfrak{N}}_1$ do not have points in common and they can be separated by the non-intersecting bounded neighborhoods $\mathfrak{G}_1$ and $\mathfrak{G}_2$:

$$\overline{\mathfrak{N}}_1\subset\mathfrak{G}_1\,,\quad \mathfrak{N}\backslash\overline{\mathfrak{N}}_1\subset\mathfrak{G}_2\,,\quad \mathfrak{G}_1\cap\mathfrak{G}_2=0\,.$$

But then, the intersection of the boundary Γ of the domain $\mathfrak{G}_1$ with the set $\mathfrak{N}$ is empty, which contradicts Theorem 5.4.

5.1.4. *Existence of characteristic vectors*

Most often we meet with equations (5.1) of the form

$$x=\mu Ax\,, \tag{5.18}$$

which it is convenient for us to write in the form

$$Ax=\lambda x\,. \tag{5.19}$$

A non-zero solution of the equation (5.18) or, what amounts to the same thing, the equation (5.19) is usually called a *characteristic vector of the operator* A. We call the corresponding value of the parameter λ a *characteristic value*, and we call the collection of characteristic values the *spectrum* of the operator A. All the terms indicated are analogous to

corresponding concepts in the theory of linear operators.
As it is known, the characteristic vectors play an important (and in many problems a decisive) role in the theory of linear operators. In the theory of non-linear operators the role of the characteristic vectors is more modest—there are only the non-zero solutions. Naturally, in the theory of non-linear operators there is no principle of superposition; a characteristic vector, multiplied by some number is not, as a rule, a characteristic vector.

Theorem 5.5. *Let Γ be the boundary of the open bounded set $\mathfrak{G} \subset E$ of which θ is an interior point. Let a positive completely continuous operator A be defined on $\Gamma \cap K$, where*

$$\inf_{x \in \Gamma \cap K} \|Ax\| > 0 . \tag{5.20}$$

Then the operator A has at least one characteristic vector x_0 on $\Gamma \cap K$:

$$Ax_0 = \lambda x_0 ,$$

to which a positive characteristic value λ corresponds.

The complete proof of this theorem will be developed in a subsequent section. Here we consider only the simplest case when Γ is the sphere $\|x\| = r$. In this case we define on the intersection T of the cone K with the sphere $\|x\| \leqslant r$ the operator $\tilde{A}$ by the equality

$$\tilde{A}x = \frac{r\left[\|x\| A\left(\frac{r}{\|x\|} x\right) - (r - \|x\|)u\right]}{\left\|\|x\| A\left(\frac{r}{\|x\|} x\right) - (r - \|x\|)u\right\|} \qquad (x \in T),$$

where u is some fixed non-zero element of K. The operator $\tilde{A}$ is evidently completely continuous and transforms T into itself. By virtue of Schauder's principle, a point $x^* \in T$ can be found such that $\tilde{A}x^* = x^*$. Since the norm of all values of the operator $\tilde{A}$ is equal to r then $\|x^*\| = r$. Therefore it follows from $\tilde{A}x^* = x^*$ that

$$Ax^* = \frac{\|Ax^*\|}{r} x^* ,$$

that is, x^* is a characteristic vector of the operator A.
Condition (1.20) is essential: if it is not satisfied, then even a linear positive completely continuous operator cannot have characteristic vectors in the cone (see ch. 2).

5.1.5. *Continuous branches of characteristic vectors.*
We shall say that the positive characteristic vectors of the operator A form a *continuous branch of length* r if the collection $\mathfrak{N}$ of all characteristic vectors lying in the cone has a non-empty intersection with the boundary Γ of every open set containing θ and which is contained together with its closure in the sphere $\|x\| < r$.
It follows from Theorem 5.5 that *the positive characteristic vectors of a positive operator A form a continuous branch of some length r if A is completely continuous and satisfies the condition $A\theta \neq \theta$.* It suffices for the proof to choose a number r such that $\|x\| < r$ $(x \in K)$ implies the inequality

$$\|Ax - A\theta\| < \|A\theta\| - \varepsilon_0 ,$$

where ε_0 is some fixed positive number. Then for the boundary Γ of an arbitrary domain lying in a sphere of radius r condition (1.20) will be satisfied:

$$\|Ax\| \geqslant \|A\theta\| - \|Ax - A\theta\| > \varepsilon_0 \qquad (x \in \Gamma \cap K) .$$

If the operator A has characteristic vectors, belonging to the cone, on the boundary Γ of an *arbitrary* bounded domain containing θ, then we shall say that these characteristic vectors form a *continuous branch of length* ∞.
In section 5.3, examples will be given of operators for which the existence of continuous branches of characteristic vectors follows from Theorem 5.5.
Finally, we shall say that the set $\mathfrak{N}$ of characteristic vectors forms a *continuous branch in the spherical shell* $r \leqslant \|x\| \leqslant R$ if the intersection of $\mathfrak{N}$ with the boundary Γ of an arbitrary domain $\mathfrak{G}$ containing the sphere $\|x\| \leqslant r$, which is contained in the sphere $\|x\| \leqslant R$ is, non-empty.

5.1.6. *Characteristic vectors of a weakly continuous operator*
The Theorem 5.5 in complete form does not generalize to weakly continuous operators but on sets $\Gamma \cap K$ of a simple nature (weakly closed) the existence of characteristic vectors is established under natural assumptions without difficulty while condition (5.20) turns out to be unnecessary.
Let, as usual, E be weakly complete, the unit sphere in E weakly compact, and the cone K allow plastering. We denote by $l(x)$ a linear uniformly positive functional on K. Let the positive operator A be weakly continuous. Then *the operator A has at least one characteristic vector on the*

intersection T_a of the cone K with every hyperplane $l(x)=a$ $(a>0)$. It suffices for the proof to consider on T_a the operator $\tilde{A}$:

$$\tilde{A}x = \frac{a(x+Ax)}{a+l(Ax)} \qquad (x\in T_a)\,.$$

This operator is weakly continuous and transforms T_a into itself. Its fixed point, which exists by virtue of Tikhonov's principle, will be a characteristic vector of the operator A.

5.2. Some topological theorems

5.2.1. *Power of a mapping*

The constructions of the present section will be based on topological concepts. Lack of space forces us to restrict ourselves to highly concise expositions. For the understanding of this section it is necessary to know only the basic facts of combinatorial topology: simplex, simplicial subdivision of a polyhedron, simplicial mapping, orientation, and so on.

For a detailed acquaintance with the basic concepts of combinatorial topology, the reader can turn to the book of P.S. Aleksandrov [1] and L. S. Pontrjagin [1]. The theory of completely continuous vector fields is presented in the book of M. A. Krasnosel'skii [5].

Let E be a finite-dimensional space in which some orientation is selected. We assume that the boundary Γ of some bounded domain $\mathfrak{G}$ allows triangulation.

Let $F(x)$ be a simplicial mapping of the polyhedron Γ onto the unit sphere $S(\|x\|=1)$. This means that Γ and S are triangulated and the continuous mapping F affinely maps every simplex of the subdivision of Γ onto some simplex of the subdivision of the sphere S.

We choose in the subdivision of the sphere S some fixed simplex Σ. We denote by t the number of those simplexes of the subdivision of Γ which under the mapping F are mapped into the simplex Σ with the retention of the orientation. Analogously, we denote by s the number of those simplexes which map into Σ with a change of orientation into the opposite orientation. The number

$$\gamma = t-s$$

is called the power of the *simplicial mapping* F of the boundary Γ of the domain $\mathfrak{G}$ onto the sphere S. This number, as it turns out, does not de-

pend on the choice of the simplex Σ in the subdivision of the sphere S. Now let $F(x)$ be an arbitrary continuous mapping Γ onto S. It turns out that for an arbitrary number $\varepsilon>0$ we can construct a simplicial mapping $F_\varepsilon(x)$ of the polyhedron Γ onto S such that

$$\|F_\varepsilon(x)-F(x)\|<\varepsilon \qquad (x\in\Gamma).$$

In this connection, the powers of all the simplicial mappings $F_\varepsilon(x)$ for sufficiently small ε are identical. We call the common power γ of all mappings F_ε the *power of the mapping* F of the boundary Γ of the domain $\mathfrak{G}$ onto the sphere S.

Two continuous mappings $\Phi(x)$ and $\Psi(x)$ of the boundary Γ onto the sphere S are called *homotopic* if a continuous function $\mathrm{X}(x;\ t)$ $(x\in\Gamma,\ 0\leqslant t\leqslant 1)$ exists with values on S such that

$$\mathrm{X}(x;\ 0)\equiv\Phi(x),\quad \mathrm{X}(x;\ 1)\equiv\Psi(x) \qquad (x\in\Gamma).$$

The basic property of the power of a mapping is expressed by the fact that *homotopic mappings* have the same power

The converse statement also holds (Hopf's theorem): *if the boundary Γ is strongly connected then from the equality of the powers of mappings follows their homotopicity.* This theorem is not used again.

5.2.2. *Rotation of a vector field*

We assume that a vector field $F(x)$ is given on Γ. This means that to every point x there is set into correspondence a vector $F(x)$ from this same space. We assume that the field $F(x)$ does not have zero vectors on Γ. Then the power γ of the mapping

$$F_1(x)=\frac{F(x)}{\|F(x)\|} \qquad (x\in\Gamma)$$

of the boundary Γ of the domain $\mathfrak{G}$ onto the unit sphere S is called a *rotation of the vector field* $F(x)$.

The two vector fields $\Phi(x)$ and $\Psi(x)$ are called *homotopic* if a vector function $\mathrm{X}(x;\ t)$ $(x\in\Gamma,\ 0\leqslant t\leqslant 1)$ exists such that

$$\mathrm{X}(x;\ 0)\equiv\Phi(x),\quad \mathrm{X}(x;\ 1)\equiv\Psi(x) \qquad (x\in\Gamma),$$

which does not take on zero values. *The rotations of homotopic vector fields are identical.*

We consider a vector field $F(x)$ given on the closed domain $\mathfrak{G}+T$. If $F(x_0)=\theta$, then we shall call x_0 a *fixed point of the field* $F(x)$ (this term becomes clear if it is assumed that $F(x)$ is a field of displacements). We shall call the

fixed point x_0 *isolated* if there are no other fixed points in some neighborhood of it. If x_0 is an isolated fixed point, then the rotation of the field $F(x)$ on all the spheres $\|x-x_0\|=\varrho$ of sufficiently small radius ϱ will be the same. We shall call this common rotation $\gamma(x_0)$ the *index* of the fixed point x_0.

We assume that the field $F(x)$ does not have zero vectors on Γ and that the domain $\mathfrak{G}$ contains a finite number of isolated fixed points $x_1, x_2, \ldots, x_k$. *The fundamental equality*

$$\gamma(\Gamma)=\gamma(x_1)+\gamma(x_2)+\ \ldots\ +\gamma(x_k)\,, \tag{5.21}$$

holds, where $\gamma(\Gamma)$ is the rotation of the vector field $F(x)$ on Γ and $\gamma(x_i)$ are the indices of the fixed points x_i.

Up to now we have assumed that the boundary Γ of the domain $\mathfrak{G}$ has certain properties. We shall now omit this assumption. Let a continuous field $F(x)$ be defined without zero vectors on Γ. This field can be extended, with retention of continuity, to the domain $\mathfrak{G}$ so that for the extended field there will only be a finite number of fixed points. We use the formula (5.21) for the definition of the rotation $\gamma(\Gamma)$ of the field F on Γ. It turns out that the rotation thus defined does not depend on the choice of the extension of the field F on $\mathfrak{G}$.

The rotation of the vector field on an arbitrary boundary Γ possesses the same properties which were shown above for the case of boundaries of a special form. In particular, homotopic fields have the same rotation.

It follows from formula (5.21) that $\gamma(\Gamma)=0$ if the field F can be extended to $\mathfrak{G}$ without zero vectors. If the domain $\mathfrak{G}$ is connected, then the converse statement holds: *a field $F(x)$ with zero rotation can be extended to $\mathfrak{G}$ without zero vectors.*

We note in conclusion one more formula which generalizes the equality (5.21).

Let $\mathfrak{G}_1, \mathfrak{G}_2, \ldots, \mathfrak{G}_k$ be mutually non-intersecting domains lying in $\mathfrak{G}$; we denote their boundaries by $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$ respectively. We assume that on $\mathfrak{G}+\Gamma$ a continuous vector field $F(x)$ is given, whose fixed points lie in the domains $\mathfrak{G}_i$ $(i=1, 2, \ldots, k)$. Then the field $F(x)$ will not have zero vectors either on Γ or on the boundaries $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$. Therefore, the rotations $\gamma(\Gamma)$ and $\gamma(\Gamma_i)$ $(i=1, 2, \ldots, k)$ will be defined. The equality

$$\gamma(\Gamma)=\gamma(\Gamma_1)+\gamma(\Gamma_2)+\ \ldots\ +\gamma(\Gamma_k) \tag{5.22}$$

holds.

We note that the rotation of the unit field $F(x)\equiv(x)$ is equal to unity if

$\theta \in \mathfrak{G}$. The rotation of the field is equal to zero if a vector u_0 can be found such that the directions of all the vectors $F(x)$ are different from the direction of u_0.

5.2.3. *Proof of Theorem 5.4 for the finite-dimensional case*
In this subsection we consider positive operators which satisfy more rigorous restrictions than those which were present in Theorem 5.4. It will be shown below how general results are obtained from this particular case. Let $A(x; \mu)$ be a positive operator which is continuous with respect to the collection of all its variables, defined on a cone K in the finite-dimensional space E. We shall assume that E is the linear hull of the cone K. The parameter μ takes values from $[m, M]$.

Lemma 5.2. *Let the conditions*

$$A(x; \mu) = u_0 \qquad (x \in K,\ \|x\| = r,\ \mu \in [m, M]) \tag{5.23}$$

and

$$A(x; \mu) = v_0 \qquad (x \in K,\ \|x\| = R,\ \mu \in [m, M]) \tag{5.24}$$

be satisfied where either

$$\|u_0\| < r < R < \|v_0\|\,, \tag{5.25}$$

or

$$r < \|u_0\|\,, \quad \|v_0\| < R\,. \tag{5.26}$$

Then the set $\mathfrak{N}$ of all solutions $x(\mu)$ of the equation

$$x = A(x; \mu) \tag{5.27}$$

for $\mu \in [m, M]$ lying in K and satisfying the inequalities

$$r < \|x(\mu)\| < R\,, \tag{5.28}$$

form a continuous branch connecting the set X_m of solutions of the equation $x = A(x; m)$ with the set X_M of solutions of the equation $x = A(x; M)$.

Proof. We denote by K_0 the cone such that every non-zero point of the cone K is an interior point of the cone K_0. The cone K_0 can be constructed since K allows plastering.
Let z_0 be a normed ($\|z_0\| = 1$) interior element of the cone K. We define on K_0 the operator A_0 by the equality

$$A_0(x; \mu) = \begin{cases} A(x; \mu)\,, & \text{if } x \in K, \\ A\left[\|x\| \dfrac{(1-t_0)z_0 + t_0 x}{\|(1-t_0)z_0 + t_0 x\|}\,; \mu\right], & \text{if } x \in K, \end{cases} \tag{5.29}$$

where t_0 is the maximal value of t for which $(1-t)z_0 + t_0 x \in K$. The con-

tinuity of the operator A_0 with respect to the collection of variables is obvious. The operator $A_0(x;\mu)$ for all values of the parameter μ does not have fixed points lying outside K.

We denote by $\mathfrak{G}$ the domain consisting of interior points x of the cone K_0, satisfying the inequalities $r<\|x\|<R$. On the boundary Γ of this domain, the vector fields

$$F(x;\mu)=x-A(x;\mu) \tag{5.30}$$

do not have fixed points for $\mu\in[m, M]$. Consequently, the fields $F(x;\mu)$ are homotopic and have the same rotation γ on Γ. It follows from elementary considerations (see M. A. Krasnosel'skii [5]) that this rotation γ is equal to 1 if condition (5.25) is satisfied.

We assume that the set $\mathfrak{N}$ does not form a continuous branch connecting X_m with X_M. Then a bounded domain $\mathfrak{G}_1\subset\mathfrak{S}$ can be found which contains X_m and does not have common points with X_M, such that on its boundary Γ_1 the equations $x=A(x;\mu)$ do not have solutions for $\mu\in[m, M]$. The vector fields (5.30) considered on Γ_1 for different μ are homotopic to one another. The rotations of these fields are equal to zero since the field $F(x; M)$ does not have zero vectors on $\mathfrak{G}_1+\Gamma_1$. It then follows from equality (5.22) that the rotation of the field $F(x; m)$ on Γ is also equal to zero. We have arrived at a contradiction.

This completes the proof of the lemma.

5.2.4. *Completion of the proof of Theorem 5.4*

Let the conditions of Theorem 5.1 be satisfied. This means, in particular, that the operator $A(x;\mu)(A(\theta;\mu)\equiv\theta)$ has a strong Fréchet derivative and a strongly asymptotic derivative with respect to the cone of the form (5.2):

$$A'(\theta;\mu)=\mu A'(\theta)\,,\quad A'(\infty;\mu)=\mu A'(\infty)\,, \tag{5.2}$$

where each of the operators $A'(\theta)$ and $A'(\infty)$ in the cone K has a unique (to within the norm) characteristic vector

$$A'(\theta)h_0=\lambda_0 h_0\,,\quad A'(\infty)g_0=\lambda_\infty g_0\,. \tag{5.3}$$

Let

$$[\mu_1,\mu_2]\subset\left(\frac{1}{\lambda_0},\frac{1}{\lambda_\infty}\right). \tag{5.15}$$

We shall show that the set $\mathfrak{N}$ of all solutions of the equation

$$x=A(x;\mu)\,, \tag{5.1}$$

corresponding to the values $\mu\in[\mu_1,\mu_2]$, forms a continuous branch, con-

necting the sets X_{μ_1} and X_{μ_2} of solutions of the equations

$$x = A(x; \mu_1)$$

and

$$x = A(x; \mu_2)$$

respectively. For definiteness we shall assume that $\lambda_0 > \lambda_\infty$.

The proof will be carried out by contradiction. The first step of the proof consists of the construction of an operator $\tilde{A}(x; \mu)$ having, on the one hand, the property that the solutions of the equation

$$x = \tilde{A}(x; \mu) \tag{5.31}$$

for $\mu \in [\mu_1, \mu_2]$ coincide with the solutions of the equation (5.1) and, on the other, that on elements $x \in K$ with large and small norms the values of the operator $\tilde{A}(x; \mu)$ are equal to two fixed elements. Moreover, with respect to the operator $\tilde{A}(x; \mu)$ some operator $P\tilde{A}(x; \mu)$ will be constructed acting in a finite-dimensional space and satisfying the conditions of Lemma 5.2; the operator $P\tilde{A}(x; \mu)$ will be constructed so that the solutions of the equation

$$x = P\tilde{A}(x; \mu)$$

do not form a continuous branch. The contradiction thus obtained will complete the proof of Theorem 5.4.

We define the operator $\tilde{A}(x; \mu)$ by an equality which is analogous to (4.53):

$$\tilde{A}(x; \mu) = \begin{cases} \left(\dfrac{2\|x\|}{r} - 1\right) A\left(\dfrac{r}{\|x\|} x; \mu\right), & \text{if } \dfrac{r}{2} \leqslant \|x\| \leqslant r, \\ A(x; \mu), & \text{if } r \leqslant \|x\| \leqslant \dfrac{1}{r}, \\ r\|x\| A\left(\dfrac{x}{r\|x\|}; \mu\right) + \left(\|x\| - \dfrac{1}{r}\right)^2 g_0, & \\ & \text{if } \dfrac{1}{r} \leqslant \|x\| \leqslant \dfrac{1}{r^2}, \\ r\|x\| \dfrac{1 - r^3\|x\|}{1 - r} A\left(\dfrac{x}{r\|x\|}; \mu\right) + \left(\dfrac{1}{r^2} - \dfrac{1}{r}\right)^2 g_0, & \\ & \text{if } \dfrac{1}{r^2} \leqslant \|x\| \leqslant \dfrac{1}{r^3}. \end{cases}$$

For a sufficiently small number r the operator $\tilde{A}(x; \mu)$ for all $\mu \in [\mu_1, \mu_2]$ has in the cone K only fixed points coinciding with the fixed points of the operator $A(x; \mu)$. We leave the proof of this fact to the reader.

The set $\mathfrak{N}$ of all solutions of the equation (5.31) for $\mu\in[\mu_1, \mu_2]$ is compact and bounded:

$$r\leqslant\|x\|\leqslant\frac{1}{r}\qquad(x\in\mathfrak{N})\,.$$

We assume that this set does not form a continuous branch joining the sets X_{μ_1} and X_{μ_2}. Then a bounded domain $\mathfrak{G}$ with the boundary Γ can be found such that $\mathfrak{N}\cap\Gamma=0$, $X_{\mu_1}\subset\mathfrak{G}$ and $X_{\mu_2}\cap\mathfrak{G}=0$. It follows from the compactness of $\mathfrak{N}$ that

$$\inf_{x\in\mathfrak{N},\,y\in\Gamma}\|x-y\|=\delta_0>0\,.$$

We denote by T_0 the set of $x\in K$ such that

$$\varrho(x,\Gamma)=\inf_{y\in\Gamma}\|x-y\|\leqslant\frac{\delta_0}{2}\,.$$

On the set T_0 the equation (5.31) for $\mu\in[\mu_1, \mu_2]$ does not have solutions. Therefore a number $\alpha>0$ can be found such that

$$\|x-\tilde{A}(x;\mu)\|>\alpha\qquad(x\in T_0,\ \mu\in[\mu_1,\mu_2])\,.\tag{5.32}$$

In fact, if a sequence $x_n\in T_0$ could be found such that $\|x_n-\tilde{A}(x_n;\mu_n)\|\to0$ where $\mu_n\in[\mu_1, \mu_2]$, then, without loss of generality, it could be assumed that μ_n would converge to some number μ^* and the elements $\tilde{A}(x_n;\mu_n)$ to some element x^*; then the elements x_n also would converge to x^* and the equality $x^*=\tilde{A}(x^*;\mu^*)$ would hold.

We denote by T_1 the set of those elements of K which do not belong to $\mathfrak{G}$ and whose norms satisfy the inequalities $r/2\leqslant\|x\|\leqslant1/r^3$; let $T_2=K\cap(\mathfrak{G}+\Gamma)$. Without loss of generality, it can be assumed that α is chosen so small that the inequalities

$$\|x-\tilde{A}(x;\mu_1)\|>\alpha\qquad(x\in T_1)\tag{5.33}$$

and

$$\|x-\tilde{A}(x;\mu_2)\|>\alpha\qquad(x\in T_2)\tag{5.34}$$

will be satisfied.

We choose in the set of all values of the operator $\tilde{A}(x;\mu)$ $(x\in K,\ r/2\leqslant\|x\|\leqslant1/r^3;\ \mu\in[\mu_1,\mu_2]$ the finite $\alpha/2$-net containing the elements θ and $v_0=[(1/r^2)-(1/r)]^2g_0$ and construct with respect to the $\alpha/2$-net the Schauder operator of projection P (see formula (4.42)). Then the operator $P\tilde{A}(x;\mu)$ will satisfy the conditions of Lemma 5.2.

By virtue of (5.33) all the solutions of the equation

$$x=P\tilde{A}(x;\mu_1)$$

lie in the domain $\mathfrak{G}$. By virtue of (5.34) no solution of the equation

$$x=P\tilde{A}(x;\mu_2)$$

lies in $\mathfrak{G}+\Gamma$. By virtue of (5.32) there are no solutions on Γ of the equation $x=P\tilde{A}(x;\mu)$ for $\mu\in[\mu_1,\mu_2]$. Thus, the solutions of the equation $x=P\tilde{A}(x;\mu)$ do not form a continuous branch, which contradicts Lemma 5.2.

This completes the proof of Theorem 5.4.

Analogues of Theorem 5.4 can be established for the cases when the conditions of Theorem 5.2 or Theorem 5.3 are satisfied.

5.2.5. *Proof of Theorem 5.5 in the finite-dimensional case*

Let Γ be the boundary of an open bounded set $\mathfrak{G}$ and the zero θ of a finite-dimensional space E be an interior point of this set. Let K be a cone in E; without loss of generality, it can be assumed that K is solid. Let a positive continuous operator A be given on $\Gamma\cap K$. We shall show that *on* $\Gamma\cap K$ *there is at least one characteristic vector of the operator* A.

We shall carry out the proof from the contrary.

If there are no characteristic vectors of A on $\Gamma\cap K$ then, in particular, A does not take on zero values on $\Gamma\cap K$ and therefore

$$\min_{x\in\Gamma\cap K}\|Ax\|=\beta>0\,.$$

We denote by T the closed convex hull of the set $A(\Gamma\cap K)$. The set T obviously does not contain the point θ.

Let the set T be k-dimensional. We draw through T a k-dimensional hyperplane E_1 (not passing, possibly, through the point θ) and we introduce in this hyperplane a system of coordinates $\xi_1, \xi_2, \ldots, \xi_k$. Then the definition of the operator A on $\Gamma\cap K$ is equivalent to the definition on $\Gamma\cap K$ of continuous functions $\xi_1(x), \xi_2(x), \ldots, \xi_k(x)$ — magnitudes of components of the vectors Ax in the system of coordinates considered. According to Uryson's theorem, every function $\xi_1(x), \xi_2(x), \ldots, \xi_k(x)$ can be extended with preservation of continuity to a closed domain $\mathfrak{G}+\Gamma$. The extension of the functions $\xi_1(x), \xi_2(x), \ldots, \xi_k(x)$ is equivalent to the construction of a continuous extension A_1 of the operator A to all $\mathfrak{G}+\Gamma$ with the set of values for the operator A_1 in the hyperplane E_1.

Let y_0 be an interior point of the set T. We denote by P the operator, defined on E_1, coinciding with the operator of the identity transformation onto T and transforming every point $y\in T(y\in E_1)$ into a point of the inter-

section of the boundary of the set T with a segment joining y and y_0. The operator P is continuous and transforms all of E_1 onto T.

We denote the operator PA_1 by $\tilde{A}$. This operator is defined on all of $\mathfrak{G}+\Gamma$ and coincides with A on $\Gamma\cap K$, where

$$\tilde{A}(\mathfrak{G}+\Gamma)\subset T\subset K$$

and

$$\min_{x\in\mathfrak{G}+\Gamma}\|\tilde{A}x\|>0\,.$$

It follows from the assumption of the absence for the operator A of characteristic vectors of $\Gamma\cap K$ that the operator $\tilde{A}$ does not have on all the boundary Γ characteristic vectors to which non-negative characteristic values correspond. Therefore, the vector fields

$$F(x;\mu)=x-\mu\tilde{A}x$$

do not have zero vectors on Γ for all $\mu\geqslant 0$. This means that all the vector fields $F(x;\mu)(\mu\geqslant 0)$ are homotopic to one another and therefore have the same rotation.

On the other hand, for $\mu=0$ the rotation of the field considered is equal to 1 and for large values of μ it is equal to zero since for large values of μ all the vectors $F(x;\mu)(x\in\Gamma)$ do not coincide in direction with the vector z_0, where z_0 is an arbitrary fixed normalized ($\|z_0\|=1$) element of K. In fact, if for an arbitrary μ one of the vectors $F(x;\mu)(x\in\Gamma)$ would coincide in direction with z_0 then a sequence $x_n\in\Gamma$ could be found such that

$$\frac{x_n-n\tilde{A}x_n}{\|x_n-n\tilde{A}x_n\|}=z_0\qquad(n=1,2,\ldots)$$

or what amounts to the same thing,

$$\frac{\frac{x_n}{n}-\tilde{A}x_n}{\left\|\frac{x_n}{n}-\tilde{A}x_n\right\|}=z_0\qquad(n=1,2,\ldots)\,.$$

A subsequence of indices n_i can be chosen such that the sequence of points x_{n_i} will converge to some point $x^*\in\Gamma$. Passing to the limit in the last equalities with respect to this subsequence of indices, we obtain

$$\tilde{A}x^*=-\|\tilde{A}x^*\|\,z_0\,,$$

that is, $Ax^*\in T$ which cannot happen according to the construction.

Thus, the vector fields $F(x;\mu)$ cannot be homotopic. The contradiction thus obtained proves the above-formulated statement.

5.2.6. *Completion of the proof of Theorem 5.5*

Now let Γ be the boundary of an open bounded set $\mathfrak{G}$ in an infinite-dimensional space E, where $\theta \in \mathfrak{G}$. We shall show that the positive completely continuous operator A has at least one characteristic vector on $\Gamma \cap K$ if the condition

$$\inf_{x \in \Gamma \cap K} \|Ax\| > 0 \tag{5.20}$$

is satisfied.

We shall carry out the proof from the contrary: we assume that there are no characteristic vectors on $\Gamma \cap K$ for the operator A and then show that the statement of Theorem 5.5 is not true for the case of finite-dimensional spaces, thus contradicting the result of the preceding subsection.

We assume that the positive completely continuous operator A satisfies the condition (5.20) but does not have on $\Gamma \cap K$ characteristic vectors to which positive characteristic values correspond. Then a number $a > 0$ can be found such that

$$\|Ax - tx\| \geqslant a \qquad (x \in \Gamma \cap K,\ 0 \leqslant t < \infty).$$

In fact, assuming the contrary, we construct a sequence of elements $x_n \in \Gamma \cap K$ $(n = 1, 2, \ldots)$ and a sequence of non-negative numbers t_n $(n = 1, 2, \ldots)$ such that

$$\|Ax_n - t_n x_n\| < \frac{1}{n} \qquad (n = 1, 2, \ldots).$$

We introduce the notation

$$r_1 = \inf_{x \in \Gamma} \|x\|, \qquad R_1 = \sup_{x \in \Gamma} \|x\|,$$

$$r_2 = \inf_{x \in \Gamma \cap K} \|Ax\|, \qquad R_2 = \sup_{x \in \Gamma \cap K} \|Ax\|.$$

It is obvious that the numbers r_1, R_1, r_2, R_2 are finite and positive. Obviously

$$t_n \leqslant \frac{\|Ax_n\| + \|Ax_n - t_n x_n\|}{\|x_n\|} < \frac{1}{r_1}\left(R_2 + \frac{1}{n}\right)$$

and

$$t_n \geqslant \frac{\|Ax_n\| - \|Ax_n - t_n x_n\|}{\|x_n\|} > \frac{1}{R_1}\left(r_2 - \frac{1}{n}\right).$$

It follows from these inequalities and from the complete continuity of the operator A that a sequence of indices n_i $(i = 1, 2, \ldots)$ could be selected

such that, obviously, the numbers t_{n_i} would converge to some positive number $t_0[(r_2/R_1) \leqslant t_0 \leqslant (R_2/r_1)]$ and the elements Ax_{n_i} would converge to some vector v_0. Then the sequence x_{n_i} converges to the element $x_0 = v_0/t_0$ since

$$\|x_{n_i} - x_0\| \leqslant \left\| x_{n_i} - \frac{1}{t_{n_i}} Ax_{n_i} \right\| + \frac{1}{t_{n_i}} \|Ax_{n_i} - v_0\| + \left\| \left(\frac{1}{t_{n_i}} - \frac{1}{t_0} \right) v_0 \right\| \leqslant$$

$$\leqslant \frac{1}{n_i t_{n_i}} + \frac{1}{t_{n_i}} \|Ax_n - v_0\| + \|v_0\| \cdot \left| \frac{1}{t_{n_i}} - \frac{1}{t_0} \right| .$$

Passing to the limit in the inequalities

$$\|Ax_{n_i} - t_{n_i} x_{n_i}\| < \frac{1}{n_i} \qquad (i = 1, 2, \ldots) ,$$

we obtain the equality $Ax_0 = t_0 x_0$, which contradicts the assumption. Thus if A satisfies condition (5.20) and does not have characteristic vectors on $\Gamma \cap K$, then

$$\inf_{x \in \Gamma \cap K,\, t \geqslant 0} \|Ax - tx\| = a > 0 .$$

We choose a finite $a/2$-net in the compact set $A(\Gamma \cap K)$. Let this $a/2$-net consist of n points $y_1, \ldots, y_n$. Since the operator A is positive, then $y_i \in K$ $(i = 1, \ldots, n)$. We define on $A(\Gamma \cap K)$ the Schauder operator (see chapter 4, section 4.3, subsection 4.3.4) of projection P by the equality

$$Py = \frac{\sum_{i=1}^{n} \varkappa_i(y) y_i}{\sum_{i=1}^{n} \varkappa_i(y)} \qquad (y \in A(\Gamma \cap K)) ,$$

where

$$\varkappa_i(y) = \begin{cases} \dfrac{a}{2} - \|y - y_i\| , & \text{if} \quad \|y - y_i\| \leqslant \dfrac{a}{2} , \\ 0, & \text{if} \quad \|y - y_i\| \geqslant \dfrac{a}{2} \end{cases} \qquad (i = 1, \ldots, n) .$$

The continuous operator P transforms every element $y \in A(\Gamma \cap K)$ into an element of the convex hull of those points of the $a/2$-net the distance to which from y is less than $a/2$. Therefore

$$\|Py - y\| < \frac{a}{2} \qquad (y \in A(\Gamma \cap K)) ,$$

from which it follows that for all $t \geqslant 0$ and $x \in \Gamma \cap K$

$$\|PAx - tx\| \geqslant \|Ax - tx\| - \|Ax - PAx\| > \frac{a}{2} . \tag{5.35}$$

Thus, the operator $\boldsymbol{PA}$ does not have characteristic vectors on $\Gamma \cap K$ to which positive characteristic values correspond.

The set of values of the operator $\boldsymbol{PA}$ lies in a finite-dimensional subspace $E_1 \subset E$ (in the linear hull of the points $y_1, \ldots, y_n$). We denote the intersection $K \cap E_1$ by K_1. The set K_1 will be a cone in E_1. Let $\mathfrak{G}_1 = \mathfrak{G} \cap E_1$ and Γ_1 be the boundary of $\mathfrak{G}_1$; obviously, $\Gamma_1 \subset \Gamma$ so that the operator $\boldsymbol{PA}$ is defined on $\Gamma_1 \cap K_1$. The operator $\boldsymbol{PA}$ satisfies the conditions of the statement proved in the preceding subsection on $\Gamma_1 \cap K_1$ (E_1 is finite-dimensional!) and therefore has at least one characteristic vector on $\Gamma_1 \cap K_1$.

We have arrived at a contradiction.

This completes the proof of the theorem.

5.2.7. *Rotation of a completely continuous vector field*

The proofs of Theorems 5.4 and 5.5 were based on the preliminary transition to the finite-dimensional case and on the subsequent use of vector fields and their properties connected with rotation. This plan allows a reversal: the theory of vector fields can be carried over to fields in infinite-dimensional spaces (the concepts of rotation, homotopicity, index of a fixed point, the formula (5.21), and so on) and thus this theory applies. In the book of M. A. Krasnosel'skii [5], the theory of the rotation of vector fields is developed and studied for the case when these fields are considered on the boundaries Γ of bounded domains; the vector fields $F(x)$ considered have the form $F(x) = x - Ax$, where A is a completely continuous operator. The applications of the theory discussed in the above-mentioned book to the investigation of operators which leave a cone K invariant are obtained directly if K is solid. For the study of operators leaving a non-solid cone K invariant, the concept of the rotation of the fields in infinite-dimensional spaces is generalized.

5.3. Operators with monotone minorants

5.3.1. *Characteristic vectors of homogeneous operators*

We call the operator $\boldsymbol{B}$ *homogeneous of degree s if*

$$\boldsymbol{B}(tx) \equiv t^s \boldsymbol{B}x \qquad (x \in K,\ t \geqslant 0)\,. \tag{5.36}$$

A linear operator can serve as an example of a homogeneous operator of the first degree.

If x_0 is a characteristic vector of the homogeneous operator B then, by virtue of (5.36), all elements of the form tx_0 will be characteristic vectors. The collection of elements of the form tx_0 form a ray. Thus, from the existence of one characteristic vector for the homogeneous operator B follows the existence of an infinite continuous branch of characteristic vectors. Therefore, for homogeneous operators, it suffices to prove the existence of one characteristic vector.

Theorem 5.6. *Let the completely continuous operator B be positive, monotone and homogeneous of degree $s \leqslant 1$. Let an element $u=v-w$ $(v, w \in K)$ exist such that $-u \bar{\in} K$ and*

$$B^p u \geqslant \alpha u , \tag{5.37}$$

where p and $\alpha > 0$ are some fixed numbers.
Then the operator B has characteristic vectors in the cone K.

Proof. We consider the operators B_n such that

$$B_n x = Bx + \frac{v}{n} \qquad (n=1, 2, \ldots) .$$

Each of these operators is completely continuous and satisfies the condition

$$\inf_{x \in K, \|x\|=1} \|B_n x\| \geqslant \inf_{y \geqslant v/n} \|y\| > 0 .$$

It follows from Theorem 5.5 that for every operator B_n there is in the cone K a normed characteristic vector x_n:

$$B_n x_n = \lambda_n x_n \qquad (\|x_n\| = 1, \quad n=1, 2, \ldots) ,$$

that is,

$$Bx_n + \frac{v}{n} = \lambda_n x_n \qquad (n=1, 2, \ldots) . \tag{5.38}$$

Without loss of generality, we can assume that the sequence of elements Bx_n converges to some element $y^* \in K$ and the numbers λ_n converge to the limit λ^* (otherwise, we could take a subsequence).
If it would turn out that $\lambda^* > 0$ (the finiteness of λ^* is obvious), then the elements x_n also would converge to some element x^* and the equality $Bx^* = \lambda^* x^*$ would follow from (5.38) after passage to the limit.
Thus, for the completion of the proof of the theorem, it suffices to show that

$$\inf_{n=1, 2, \ldots} \lambda_n > 0 . \tag{5.39}$$

It follows from (5.38) that

$$Bx_n \leqslant \lambda_n x_n \qquad (n=1, 2, \ldots).$$

Therefore

$$B^p x_n \leqslant B^{p-1}(\lambda_n x_n) \leqslant B^{p-2}(\lambda_n{}^{1+s} x_n) \leqslant \ldots \leqslant \lambda_n{}^{1+s+\ldots+s^{p-1}} x_n. \tag{5.40}$$

It also follows from (5.38) that

$$x_n \geqslant \frac{1}{n\lambda_n} v.$$

Therefore a number $t_n > 0$ can be found such that $x_n \geqslant t_n v$ and $x_n \geqslant tv$ for $t > t_n$. Then

$$B^p x_n \geqslant B^p(t_n v) = t_n{}^{s^p} B^p v \geqslant t_n{}^{s^p} B^p u$$

and by virtue of (5.37)

$$B^p x_n \geqslant \alpha t_n{}^{s^p} u.$$

Combining the last inequality with (5.40), we come to the inequality

$$\lambda_n{}^{1+s+\ldots+s^{p-1}} x_n \geqslant \alpha t_n{}^{s^p} u.$$

It follows from the maximality of t_n that

$$\frac{\alpha t_n{}^{s^p}}{\lambda_n{}^{1+s+\ldots+s^{p-1}}} \leqslant t_n \qquad (n=1, 2, \ldots),$$

that is,

$$\lambda_n{}^{1+s+\ldots+s^p} \geqslant \frac{\alpha}{t_n{}^{1-s^p}} \qquad (n=1, 2, \ldots). \tag{5.41}$$

In order to obtain (5.39) from (5.41), it is necessary to show that the numbers t_n are bounded above. But this is obvious since in the opposite case $-v \in K$ would follow from $x_n \geqslant t_n v$.
This completes the proof of the theorem.

An estimate for the characteristic value λ^* can be obtained from (5.41). If, for example, $s=1$ then it follows from (5.41) that

$$\lambda^* \geqslant \sqrt[p]{\alpha}. \tag{5.42}$$

If $0<s<1$, then

$$\lambda^* \geqslant t_0{}^{s-1} \alpha^{(1-s)/(1-s^p)}, \tag{5.43}$$

where t_0 denotes the greatest number such that the inequality $x \geqslant t_0 v$ holds for at least one normalized element x of the cone K.

5.3.2. *Continuous branch of characteristic vectors for an operator with a monotone homogeneous minorant*

We recall that the operator B is a minorant for the operator A on the set T if

$$Ax \geqslant Bx \qquad (x \in T). \tag{5.44}$$

As usual, we shall denote by K_r the intersection of the cone K with the sphere $\|x\| \leqslant r$.

Theorem 5.7. *Let the completely continuous operator A have a minorant B on K_r satisfying the conditions of Theorem* 5.6★.
Then A has a continuous branch of length r of characteristic vectors in the cone K.

Proof. As in the proof of Theorem 5.6, we construct the operators A_n:

$$A_n x = Ax + \frac{v}{n} \qquad (n = 1, 2, \ldots).$$

Let Γ be the boundary of some domain $\mathfrak{G}$ contained in the sphere $\|x\| < r$ of which θ is an interior point. By virtue of Theorem 5.5, every operator A_n has at least one characteristic vector x_n on $\Gamma \cap K$:

$$A_n x_n = \lambda_n x_n \qquad (x_n \in \Gamma \cap K, \quad \lambda_n > 0),$$

that is,

$$Ax_n + \frac{v}{n} = \lambda_n x_n \qquad (n = 1, 2, \ldots). \tag{5.45}$$

Without loss of generality, we can assume that the elements Ax_n converge to some y^* and the numbers λ_n converge to the limit λ^*. As in the proof of Theorem 5.6, it remains to show that the numbers λ_n are bounded below by a positive constant.
It follows from (5.45) that

$$Bx_n + \frac{v}{n} \leqslant \lambda_n x_n \qquad (n = 1, 2, \ldots),$$

that is

$$Bx_n \leqslant \lambda_n x_n \qquad (n = 1, 2, \ldots)$$

and

$$x_n \geqslant \frac{1}{n\lambda_n} v \qquad (n = 1, 2, \ldots).$$

★ For the proof of Theorem 5.7, complete continuity of the operator B is not used.

Namely, from the last inequalities in the proof of Theorem 5.6 there were obtained the inequalities

$$\lambda_n^{1+\varepsilon+\ldots+\varepsilon^{p-1}} \geqslant \frac{\alpha}{t_n^{1-\varepsilon^p}} \qquad (n=1, 2, \ldots),$$

where t_n are numbers such that $x_n \geqslant t_n v$ and $x_n \not\geqslant tv$ for $t > t_n$. The boundedness of the numbers t_n follows from the boundedness above of the numbers λ_n.
This completes the proof of the theorem.

If, in particular, there is a monotone minorant for the operator A on the entire cone K, then the characteristic vectors of the operator A in the cone K form a continuous branch of infinite length.

5.3.3. *Fundamental theorem*
If $p=1$ in condition (5.37), then for the operator B fewer requirements can be laid claim to than in Theorems 5.6 and 5.7.

Theorem 5.8. *Let the positive completely continuous operator A have a positive monotone minorant B on K_r. Let an element $u=v-w(v, w\in K)$ exist such that $-u\overline{\in}K$ and*

$$B(tu)\geqslant \alpha tu \qquad (0\leqslant t\leqslant \gamma), \tag{5.46}$$

where γ is a number such that it follows from $x\geqslant tu(t\geqslant 0)$ and from $\|x\|\leqslant r$ that $t\leqslant\gamma$.
Then A has a continuous branch of length r of characteristic vectors in the cone K.

Proof. As in the preceding theorem we establish the existence of elements $x_n\in\Gamma$ such that

$$Ax_n + \frac{v}{n} = \lambda_n x_n \qquad (n=1, 2, \ldots).$$

From these equalities follow the inequalities $Bx_n\leqslant\lambda_n x_n$ and the existence of numbers $t_n>0$ such that $x_n\geqslant t_n u$ and $x_n\not\geqslant tu$ for $t>t_n$. Then

$$\lambda_n x_n\geqslant Bx_n\geqslant B(t_n u)\geqslant\alpha t_n u,$$

from which it follows that $\lambda_n\geqslant\alpha$. The characteristic vector of the operator A can be constructed as the limit of the elements x_n.
This completes the proof of the theorem.

5.3.4. *Principle of topological extension*
For the proof of existence theorems the so-called extension principle is

often applied. It consists in the following.

It is required to prove the existence of solutions for the equation

$$x = Ax \tag{5.47}$$

where A is an operator acting in a Banach space E. An auxiliary operator $A(x; \mu)$ is constructed depending on the parameter $\mu \in [0, 1]$. The parameter μ is introduced in such a way that the equation $x = A(x; 0)$ is easy to solve (or in order that the existence theorem be easily proved for it). The equation $x = A(x; 1)$ must coincide with the equation (5.47). The solution of equation (5.47) can then be obtained, observing how the solution $x(\mu)$ of

$$x = A(x; \mu) \tag{5.48}$$

is changed. In order to observe the solution $x(\mu)$, usually rigorous restrictions must be placed on the operator considered. In particular, the uniqueness (in some domain) of the solution of equation (5.48) has to be guaranteed.

The plan presented below is close to the principle of extension; however, in a number of cases it has essential advantages. For example, it is assumed below that the solutions of equation (5.48) are isolated. We restrict ourselves to the description of the principle of topological extension applied to the problem of the solutions lying in a cone K.

Thus, let it be required to prove that the equation (5.47) has a solution $x \in K$. As above, we introduce in the equation a parameter μ, which can assume different numerical values. The parameter μ is introduced in order that the solutions of the equation (5.48) will form in K a continuous branch $\mathfrak{N}$ of infinite length if the solutions corresponding to all non-negative μ are considered.

We assume that we can succeed in proving that the values of parameter μ approach some limit μ_0, when the norms of the solutions $x(\mu)$ of equation (5.48) approach zero. Analogously, let the values of μ approach μ_∞ when the norms of the solutions $x(\mu)$ increase without limit. Then it can be expected that there are solutions in the continuous branch $\mathfrak{N}$ corresponding to all values of $\mu \in (\mu_0, \mu_\infty)$.

Finally, let the equation (5.48) reduce to equation (5.47) for $\mu = 1$. Then the existence of solutions of the equation (5.47) will be proved if

$$1 \in (\mu_0, \mu_\infty) . \tag{5.49}$$

Carrying out the above-described principle of topological extension con-

sists in several steps. One must know how to construct the operator $A(x;\mu)$—it is difficult to indicate general recipes here; in many cases it suffices to set $A(x;\mu)=\mu Ax$. It is necessary to show that the solutions of the equation (5.48) form a continuous branch; if $A(x;\mu)=\mu Ax$, then here we can make use of Theorem 5.7 or Theorem 5.8. Moreover, it is necessary to find numbers μ_0 and μ_∞; general methods for discovering them in a particular but important case are shown in the following subsection. Finally, it is necessary to know that the values of the parameter μ for which the equation (5.48) has a solution in the cone K completely fill the interval (μ_0, μ_∞); it is convenient here to use the following theorem:

Theorem 5.9. *Let the operator $A(x;\mu)$ $(0\leqslant\mu<\infty)$ be completely continuous and let all non-zero solutions of the equation* (5.48) *form in K a continuous locally compact branch $\mathfrak{N}$ of infinite length. Let $\mu\to\mu_0$ and $\mu\to\mu_\infty$, when $\|x(\mu)\|\to 0$ and $\|x(\mu)\|\to\infty$, respectively, where $x(\mu)\in\mathfrak{N}$.*
Then for every $\mu^*\in(\mu_0,\mu_\infty)$ the equation (5.48) *has at least one solution $x(\mu^*)\in\mathfrak{N}$.*

Proof. We assume that the statement of the lemma is not true, that is, $\mu^*\in(\mu_0,\mu_\infty)$ but the equation

$$x=A(x;\mu^*)$$

does not have any non-zero solution in the cone K.
The number μ^* partitions the straight line $(-\infty,\infty)$ of real numbers into two parts. One of them, say J_1, either contains μ_∞ or μ_∞ is an endpoint of it (in this case $\mu_\infty=\infty$). The second part, J_2, contains μ_0.
We denote by $\mathfrak{N}_1$ the set of those non-zero solutions of the equation (5.48) to which the values of the parameter μ of J_1 correspond. We denote by $\mathfrak{N}_2$ the set of the remaining solutions of the equation (5.48) to which the element θ is adjoined. The sets $\mathfrak{N}_1$ and $\mathfrak{N}_2$ contain all the elements of the continuous branch $\mathfrak{N}$.
The sets $\mathfrak{N}_1$ and $\mathfrak{N}_2$ evidently are closed and do not intersect. In this connection, the set $\mathfrak{N}_1$ is found at a positive distance from the point θ and the set $\mathfrak{N}_2$ is bounded.
The distance between the sets $\mathfrak{N}_1$ and $\mathfrak{N}_2$ is positive. Assume the contrary, we can find sequences $x_n\in\mathfrak{N}_1$ and $y_n\in\mathfrak{N}_2$ such that

$$x_n=A(x_n;\mu_n)\,,\quad y_n=A(y_n;\nu_n)\,,$$

where

$$\lim_{n\to\infty} \|x_n - y_n\| = 0 .$$

The sequence y_n is bounded; consequently, the sequence x_n is bounded. One of the numerical sequences μ_n or ν_n is bounded; without loss of generality, it can be assumed to be convergent. Suppose, for definiteness, that the sequence ν_n converges. It then follows from the complete continuity of the operator $A(x;\mu)$ that the sequence y_n can be assumed to be convergent to some element x^*. Then the sequence x_n converges to x^*. This means that $x^* \in \mathfrak{N}_1$ and $x^* \in \mathfrak{N}_2$. We have arrived at a contradiction.
Thus

$$\rho(\mathfrak{N}_1, \mathfrak{N}_2) = \inf_{x\in\mathfrak{N}_1,\, y\in\mathfrak{N}_2} \|x-y\| = \alpha > 0 . \tag{5.50}$$

We denote by $\mathfrak{G}$ the bounded open set formed by the union of all the spheres of radius $\alpha/2$ with centers in the points of the set $\mathfrak{N}_2$. We denote the boundary of the set $\mathfrak{G}$ by Γ.
It is obvious that the intersection $\Gamma \cap \mathfrak{N}_2$ is empty. By virtue of (5.50) the intersection $\Gamma \cap \mathfrak{N}_1$ is empty. This means that there is no solution of the equation (5.48) on Γ for any value of the parameter μ. This means that $\mathfrak{N}$ is not a continuous branch of infinite length.
We have arrived at a contradiction.
This completes the proof of the theorem.

5.3.5. *Bifurcational values of the parameter*
In this subsection we consider the question concerning which limits the positive characteristic values of the positive completely continuous operator $A(A\theta = \theta)$ approach when the norms of the characteristic vectors approach zero or increase without limit. Such limit values are called *bifurcational values.*
We shall assume that the operator A has a strong Fréchet derivative $A'(\theta)$ with respect to a cone and a strong asymptotic derivative $A'(\infty)$ with respect to a cone. We assume that the existence of a continuous branch of infinite length of characteristic vectors has been proved.
Let

$$Ax_n = \lambda_n x_n \qquad (x_n \in K, \quad n = 1, 2, \ldots) \tag{5.51}$$

and $\|x_n\| \to 0$. Then, without loss of generality, we can assume that the numbers λ_n approach some numerical limit λ_0 and the elements $A'(\theta)[x_n/\|x_n\|]$ approach some element $v_0 \in K$. We rewrite the equalities (5.51) in the

form

$$\frac{Ax_n - A'(\theta)x_n}{\|x_n\|} + A'(\theta)\frac{x_n}{\|x_n\|} = \lambda_n \frac{x_n}{\|x_n\|} \qquad (n=1, 2, \ldots)$$

and pass to the limit as $n\to\infty$. It is clear from the limit equality that two variants are possible: either $\lambda_0=0$ and $v_0=\theta$ or $\lambda_0>0$ and the elements $x_n/\|x_n\|$ converge to some element $u_0\in K$, where $A'(\theta)u_0=\lambda u_0$, that is, λ_0 is a positive characteristic value of the operator $A'(\theta)$.

It is proved analogously that for $\|x_n\|\to\infty$ the characteristic values (if they converge) approach either zero or a positive characteristic value of the operator $A'(\infty)$.

We assume that the operator A being studied has a linear monotone minorant B. We denote by the collection $\mathfrak{N}$ of all those characteristic vectors the existence of which follows from the proof of Theorem 5.7. These characteristic vectors form a continuous branch and to these characteristic vectors correspond characteristic values which are uniformly bounded below by some positive number (the number $\sqrt[p]{\alpha}$). Therefore, for such operators, the numbers λ_n in the equalities (5.51) do not approach zero when $\|x_n\|\to 0$ or $\|x_n\|\to\infty$. Consequently for operators with a linear minorant the limit values of the parameter λ coincide with the positive characteristic values of the operators $A'(\theta)$ and $A'(\infty)$.

If it is assumed that each of the operators $A'(\theta)$ and $A'(\infty)$ has a unique positive characteristic value λ_0 and λ_∞, respectively, then it can be concluded from Theorem 5.9 that all the numbers λ of the interval $(\lambda_0, \lambda_\infty)$ are positive characteristic values. In particular, if $1\in(\lambda_0, \lambda_\infty)$, then the equation $x=Ax$ has at least one non-zero solution in the cone K. The last statement was obtained in chapter 4 by another method.

5.3.6. *Estimates of characteristic values*

It is not difficult to obtain exact estimates of characteristic values if the operator has linear minorants or majorants which are u_0-positive operators.

Theorem 5.10. *Let B_- and B_+ be two linear u_0-positive operators. Let λ_- and λ_+ be the unique positive characteristic values of the operators B_- and B_+. Let A_- be a non-linear positive operator and*

$$Ax_0=\lambda_0 x_0\,. \tag{5.52}$$

Then from the condition

$$B_-x\leqslant Ax \qquad (x\in K) \tag{5.53}$$

it follows that

$$\lambda_0 \geqslant \lambda_- , \tag{5.54}$$

and from the condition

$$Ax \leqslant B_+ x \qquad (x \in K) \tag{5.55}$$

it follows that

$$\lambda_0 \leqslant \lambda_+ . \tag{5.56}$$

Proof. Let

$$B_- h = \lambda_- h_0 \qquad (h_0 \in K,\ h_0 \neq \theta) .$$

By virtue of Lemma 2.1, the operator B_- is h_0-positive. Therefore a natural number p and positive numbers α and β can be found such that

$$\alpha h_0 \leqslant B_-^p x_0 \leqslant \beta h_0 .$$

Then

$$x_0 = \frac{1}{\lambda_0} A x_0 \geqslant \frac{1}{\lambda_0} B_- x_0 = \frac{1}{\lambda_0^2} B_- A x_0 \geqslant \frac{1}{\lambda_0^2} B_-^2 x_0 \geqslant \ldots \geqslant$$
$$\geqslant \frac{1}{\lambda_0^p} B_-^p x_0 \geqslant \frac{\alpha}{\lambda_0^p} h_0 .$$

Therefore a number $t_0 > 0$ can be found such that

$$x_0 \geqslant t_0 h_0 , \tag{5.57}$$

but $x_0 \not\geqslant t h_0$ for $t > t_0$.
It follows from (5.57) that

$$x_0 = \frac{1}{\lambda_0} A x_0 \geqslant \frac{1}{\lambda_0} B_- x_0 \geqslant \frac{t_0}{\lambda_0} B h_0 = \frac{\lambda_-}{\lambda_0} t_0 h_0 ,$$

and (5.54) follows from the maximality of t_0.
Analogously, if condition (5.55) is satisfied and

$$B_+ g_0 = \lambda_+ g_0 \qquad (g_0 \in K,\ g_0 \neq \theta) ,$$

then the operator B_+ is g_0-positive. Let

$$\alpha_1 g_0 \leqslant B_+ x_0 \leqslant \beta_1 g_0 ,$$

where α_1 and β_1 are positive. Then

$$x_0 = \frac{1}{\lambda_0} A x_0 \leqslant \frac{1}{\lambda_0} B_+ x_0 = \frac{1}{\lambda_0^2} B_+ A x_0 \leqslant \frac{1}{\lambda_0^2} B_+^2 x_0 \leqslant \ldots \leqslant$$
$$\leqslant \frac{1}{\lambda_0^q} B_+^q x_0 \leqslant \frac{\beta_1}{\lambda_0^q} g_0 .$$

Therefore a positive number t_0 can be found such that

$$x_0 \leqslant t_0 g_0 \tag{5.58}$$

and $x_0 \not\leqslant t g_0$ for $t < t_0$. It follows from (5.58) that

$$x_0 = \frac{1}{\lambda_0} A x_0 \leqslant \frac{1}{\lambda_0} B_+ x_0 \leqslant \frac{t_0}{\lambda_0} B_+ g_0 = \frac{\lambda_+}{\lambda_0} t_0 g_0 ,$$

and (5.56) follows from the minimality of t_0.
This completes the proof of the theorem.

The statement of the theorem, as is seen from the proof, remains valid if the assumption about the u_0-positiveness of the operators B_- and B_+ is discarded and replaced by the following less convenient but less restricting assumption. It suffices to assume that the operator B_- is h_0-bounded below and B_+ is g_0-bounded above, where h_0 and g_0 are positive characteristic vectors of the operators B_- and B_+, respectively, which correspond to the positive characteristic values λ_- and λ_+.

Chapter 6

EQUATIONS WITH CONCAVE OPERATORS

Classes of equations with positive operators are chosen for which the uniqueness theorem for a non-zero positive solution is valid. The convergence of the successive approximations to this unique solution is studied.

6.1. Uniqueness of a positive solution and the convergence of successive approximations

6.1.1. *Concave operators*

We shall call the operator A, acting in the space E with the cone K, *concave* if there exists a non-zero element $u_0 \in K$ such that for an arbitrary non-zero $x \in K$ the inequalities

$$\alpha u_0 \leqslant Ax \leqslant \beta u_0 , \tag{6.1}$$

where α and β are positive, are valid, and if for every $x \in K$ such that $\alpha_1(x)u_0 \leqslant x \leqslant \beta_1(x)u_0$ $(\alpha_1(x), \beta_1(x) > 0)$ the relations

$$A(tx) \geqslant tAx \qquad (0 \leqslant t \leqslant 1) \tag{6.2}$$

are valid.

In the case of a one-dimensional space, a concave operator is a function $f(x)$ defined for non-negative values of the independent variable whose graph has the property that on every segment $[0, x_0]$ this graph does not have points situated below the straight line passing through the origin and the point $\{x_0, f(x_0)\}$. A concave operator is not necessarily continuous and not necessarily monotone.

The operator $Ax \equiv u_0$ is the simplest example of a concave operator.

If the operator A is concave, then the operator cA will be concave where c is an arbitrary positive number. If A is concave and monotone, then the operators A^2, A^3 and so on will possess these properties.

In the definition of a concave operator some element u_0 appears. We assume that two concave operators A and B are given; the same element u_0

appears in the definition of concavity of these two operators. Then the operator $A+B$ will be concave.
We shall not go into the proof of the formulated statements since these proofs are obvious.

6.1.2. *Characteristic vectors corresponding to distinct characteristic values*
In a number of theorems stated below, characteristic vectors of a concave operator A are studied. For the theorems of this section *it is assumed that corresponding characteristic vectors exist*; the question concerning the existence of characteristic vectors is considered later.

Theorem 6.1. *Let the operator A be concave and monotone. Let x_1 and x_2 be non-zero positive solutions of the equation*

$$Ax=\lambda x\,, \tag{6.3}$$

corresponding to the distinct values λ_1 and λ_2 of the parameter λ:

$$Ax_1=\lambda_1 x_1, \quad Ax_2=\lambda_2 x_2\,.$$

Finally, let

$$\lambda_1<\lambda_2\,. \tag{6.4}$$

Then $x_1\geqslant x_2$.

Proof. We assume the contrary. Let $x_1\gtrless x_2$. We denote by t_0 a number such that $x_1\geqslant t_0 x_2$ and $x_1\geqslant t x_2$ for $t>t_0$. Obviously, $0\leqslant t_0<1$.
It follows from the definition of the concavity of the operator A that

$$\alpha_1 u_0\leqslant Ax_1\leqslant\beta_1 u_0\,, \quad \alpha_2 u_0\leqslant Ax_2\leqslant\beta_2 u_0\,,$$

where $\alpha_1,\alpha_2, \beta_1, \beta_2>0$. Therefore

$$x_1=\frac{1}{\lambda_1}Ax_1\geqslant\frac{\alpha_1}{\lambda_1}u_0\geqslant\frac{\alpha_1}{\lambda_1\beta_2}Ax_2=\frac{\alpha_1\lambda_2}{\lambda_1\beta_2}x_2\,.$$

Consequently, $t_0>0$.
It then follows from the monotoneity and concavity of the operator A that

$$x_1=\frac{1}{\lambda_1}Ax_1\geqslant\frac{1}{\lambda_1}A(t_0x_2)\geqslant\frac{t_0}{\lambda_1}Ax_2=\frac{\lambda_2}{\lambda_1}t_0x_2\,.$$

It follows from the maximality of t_0 that $\lambda_2\leqslant\lambda_1$ and this contradicts (6.4). This completes the proof of the theorem.

Let the conditions of Theorem 6.1 be satisfied. We consider the conical segment $\langle x_2, x_1\rangle$. Let $\lambda\in(\lambda_1, \lambda_2)$. The operator $A(\lambda)=(1/\lambda)A$ transforms

this segment into itself since $A(\lambda)$ is monotone and

$$A(\lambda)x_2 = \frac{1}{\lambda}Ax_2 = \frac{\lambda_2}{\lambda}x_2 \geqslant x_2$$

and

$$A(\lambda)x_1 = \frac{1}{\lambda}Ax_1 = \frac{\lambda_1}{\lambda}x_1 \leqslant x_1 .$$

This allows one to apply Theorem 4.1 or Theorem 4.2 for the proof of the existence of fixed points for the operator $A(\lambda)$ (that is, for the proof of the existence of solutions for the equation (6.3)). In particular, from Theorem 4.1 follows

Theorem 6.2. *Let the equation* (6.3) *with a concave and monotone operator* A *have non-zero positive solutions* x_1 *and* x_2 *for positive values* λ_1 *and* λ_2 *of the parameter* λ.
Let, also, one of the following conditions be satisfied:
6.2(a) *The cone* K *is strongly minihedral.*
6.2(b) *The cone* K *is regular, the operator* A *is continuous.*
6.2(c) *The cone* K *is normal, the operator* A *is completely continuous.*
Then the equation (6.3) *has non-zero positive solutions for all* $\lambda \in (\lambda_1, \lambda_2)$.

6.1.3. *Uniqueness theorem*

We call the concave operator A u_0-*concave* if condition (6.2) is replaced by a more stringent restriction: for every positive number $t_0 \in (0, 1)$ a number $\eta = \eta(x; t_0) > 0$ can be found such that

$$A(t_0 x) \geqslant (1+\eta)t_0 Ax . \tag{6.5}$$

Here, as in condition (6.2), $x \in K$ are considered such that

$$\alpha_1(x)u_0 \leqslant x \leqslant \beta_1(x)u_0 \qquad (\alpha_1(x) > 0 , \quad \beta_1(x) > 0) .$$

The operator $Ax \equiv u_0$ is a u_0-concave oparator. If A is a u_0-concave operator, then all the operators cA for $c > 0$ are u_0-concave.
If the operator B is concave and the element u_0 appears in the definition of its concavity, then the u_0-concavity of the operator $A + B$ follows from the u_0-concavity of the operator A. We now prove this simple fact.
Let the inequality (6.5) be satisfied and let

$$\alpha u_0 \leqslant Ax \leqslant \beta u_0, \quad \alpha_1 u_0 \leqslant Bx \leqslant \beta_1 u_0 .$$

We set

$$\eta_1 = \frac{\alpha\eta}{\alpha + \beta_1} .$$

Then

$$(1+\eta)Ax+Bx\geqslant$$

$$\geqslant(1+\eta_1)Ax+\left(\eta-\frac{\alpha\eta}{\alpha+\beta_1}\right)\alpha u_0+(1+\eta_1)Bx-\frac{\alpha\eta}{\alpha+\beta_1}\beta_1 u_0=$$

$$=(1+\eta_1)(Ax+Bx),$$

that is,

$$(A+B)(t_0x)\geqslant(1+\eta)t_0Ax+t_0Bx\geqslant(1+\eta_1)t_0(A+B)x.$$

Theorem 6.3. *If the operator A is u_0-concave and monotone, then the equation* (6.3) *does not have two distinct non-zero solutions in the cone for some value of the parameter λ.*

Proof. We assume that

$$Ax_1=\lambda x_1,\quad Ax_2=\lambda x_2,$$

where x_1 and x_2 are distinct non-zero elements of K. Without loss of generality, it can be assumed that $x_1\geqq x_2$. As in the proof of Theorem 6.1, we introduce a maximal number t_0 such that $x_1\geqslant t_0x_2$ and $x_1\geqq tx_2$ for $t>t_0$ and show that $0<t_0<t$.
It follows from the u_0-concavity of the operator A that

$$A(t_0x_2)\geqslant(1+\eta)t_0Ax_2,$$

where $\eta>0$. But then the inequality

$$x_1=\frac{1}{\lambda}Ax_1\geqslant\frac{1}{\lambda}A(t_0x_2)\geqslant\frac{(1+\eta)t_0}{\lambda}Ax_2=(1+\eta)t_0x_2$$

follows from the monotoneity of the operator A.
The last inequality contradicts the maximality of t_0.
This completes proof of the theorem.

We call the operator A *u_0-monotone* if it is concave, if $A(tx)\neq tAx$ for $t\neq0$, 1 and for every $x\in K$ such that $x\geqslant\gamma u_0$ $(\gamma>0)$ and if, finally, it follows from $x\geqslant y$ and $x\neq y$ that $Ax\geqslant Ay+\varepsilon_0u_0$, where $\varepsilon_0=\varepsilon_0(x,y)>0$. If A is u_0-monotone, then all the operators cA, where $c>0$, are u_0-monotone.

Theorem 6.4. *If the operator A is u_0-monotone, then the equation* (6.3) *does not have two distinct non-zero solutions in the cone for some value of the parameter λ.*

Proof. We assume the contrary and (as in the proof of Theorem 6.3) denote by t_0 a number from (0, 1) such that $x_1\geqslant t_0x_2$ and $x_1\geqq tx_2$ for $t>t_0$,

where

$$Ax_1=\lambda x_1\,,\quad Ax_2=\lambda x_2\,.$$

It would follow from the equality $x_1=t_0x_2$ that

$$x_1=\frac{1}{\lambda}Ax_1=\frac{1}{\lambda}A(t_0x_2)\neq\frac{t_0}{\lambda}Ax_2=t_0x_2=x_1\,.$$

Therefore, $x_1\neq t_0x_2$ and

$$Ax_1\geqslant A(t_0x_2)+\varepsilon u_0\geqslant t_0Ax_2+\varepsilon u_0\,.$$

Let $Ax_2\leqslant\beta u_0$. Then it follows from the preceding inequality that

$$Ax_1\geqslant\left(t_0+\frac{\varepsilon}{\beta}\right)Ax_2\,,$$

that is,

$$x_1\geqslant\left(t_0+\frac{\varepsilon}{\beta}\right)x_2\,,$$

and this inequality contradicts the maximality of t_0.
This completes the proof of the theorem.

6.1.4. *The square of a concave operator*
Every solution of the equation $Ax=x$ is simultaneously a solution of all the equations $A^2x=x$, $A^3x=x$, and so on. Therefore the proof of the uniqueness of the solution of the equation $Ax=x$ is equivalent to the proof of the uniqueness of the solution for the equation $A^nx=x$ for some n. It suffices, for example, to prove that the operator A^n is u_0-concave.
In this manner, it is easy to obtain a new proof of Theorem 6.4 since the square of a u_0-monotone operator is a u_0-concave operator. We prove a more general statement.

Theorem 6.5. *Let the operator A be concave and monotone. Let the inequality*

$$A(t_0x)\neq t_0Ax\qquad(0<t_0<1)\tag{6.6}$$

be satisfied for every x such that $\alpha(x)u_0\leqslant x\leqslant\beta(x)u_0$ $(\alpha(x),\,\beta(x)>0)$.
Let it follow, for an arbitrary pair of non-zero elements v and w $(\alpha_1u_0\leqslant v\leqslant\beta_1u_0,\ \alpha_2u_0\leqslant w\leqslant\beta_2u_0,\ \alpha_1,\alpha_2,\beta_1,\beta_2>0)$, from

$$t_0v\leqslant w\leqslant v\qquad(t_0v\neq w,\ w\neq v,\ t_0>0)\tag{6.7}$$

that

$$Aw\geqslant t_0Av+\varepsilon_0u_0\,,\tag{6.8}$$

where $\varepsilon_0 > 0$.
Then A^2 *is a* u_0*-concave operator.*

Proof. Let x be an arbitrary element satisfying the conditions

$$\alpha u_0 \leqslant x \leqslant \beta u_0 \qquad (\alpha, \beta > 0),$$

and let $0 < t_0 < 1$. It follows from the concavity and monotoneity of the operator A that

$$Ax \geqslant A(t_0 x) \geqslant t_0 Ax, \tag{6.9}$$

where the inequality (6.6) is satisfied.
If $A(t_0 x) = Ax$, then this equality can be considered as condition (6.5).
If $A(t_0 x) \neq Ax$, then (6.9) can be considered as the inequality (6.7). The inequality (6.8) is then rewritten in the form

$$A^2(t_0 x) \geqslant t_0 A^2 x + \varepsilon u_0. \tag{6.10}$$

Let $A^2 x \leqslant \beta u_0 (\beta > 0)$. It then follows from the inequality

$$A^2(t_0 x) \geqslant \left(t_0 + \frac{\varepsilon}{\beta}\right) A^2 x,$$

which is condition (6.5).
This completes the proof of the theorem.

It is not difficult to see that u_0-monotone operators and, of course, u_0-concave operators satisfy the conditions of Theorem 6.5.

6.1.5. *Criteria for concavity of an operator in terms of the properties of the derivatives*
We assume that the operator A is differentiable with respect to a cone and, moreover, that the value of the operator A at every point $x \in K$ is representable in the form

$$Ax = A\theta + \int_0^1 A'(\sigma x) x \, d\sigma. \tag{6.11}$$

Analogously, as the concavity of a common scalar function follows from the fact that the derivative is non-increasing, property (6.2) of the operator A follows from condition

$$A'(\sigma_1 x) x \leqslant A'(\sigma_2 x) x \qquad (x \in K,\ 0 \leqslant \sigma_2 \leqslant \sigma_1). \tag{6.12}$$

In fact, it follows from (6.12) and (6.11) that for $t \in (0,1)$

$$A(tx)=A\theta+t\int_0^1 A'(\sigma tx)x\,d\sigma=$$

$$=A\theta+\int_0^t A'(sx)x\,ds\geqslant A\theta+\int_0^t A'\left(\frac{s}{t}x\right)x\,ds=$$

$$=A\theta+t\int_0^1 A'(\sigma x)x\,d\sigma,$$

that is,

$$A(tx)\geqslant(1-t)A\theta+tAx. \tag{6.13}$$

Condition (6.1) of the concavity of the operator A is satisfied if $A\theta$ is u_0-measurable and the derivatives $A'(x)$ $(x\in K)$ satisfy the condition

$$\alpha(x)u_0\leqslant A'(x)x\leqslant\beta(x)u_0, \tag{6.14}$$

where $\alpha(x)$, $\beta(x)>0$ for $x\neq 0$. The proof is obvious: by virtue of (6.11),

$$Ax\leqslant A\theta+u_0\int_0^1\beta(\sigma x)\,d\sigma\leqslant\beta u_0$$

and

$$Ax\geqslant A\theta+u_0\int_0^1\alpha(\sigma x)\,d\sigma\geqslant\alpha u_0.$$

If $A\theta\geqslant\alpha_0 u_0$ where $\alpha_0>0$, then the operator A is u_0-concave. This follows from (6.13): for $t_0\in(0,1)$,

$$A(t_0x)\geqslant(1-t_0)A\theta+t_0Ax\geqslant(1-t_0)\alpha_0u_0+t_0Ax\geqslant$$

$$\geqslant\left[t_0+(1-t_0)\frac{\alpha_0}{\beta}\right]Ax.$$

Less restrictive properties of the derivative can be given which guarantee the concavity or u_0-concavity of the operator A. It is more complicated to describe these and we do not go into them. It is not difficult to write out the properties of the derivatives for which the operator A is u_0-monotone or satisfies the conditions of Theorem 6.5.

6.1.6. *Fundamental theorem concerning the convergence of successive approximations*

In section 4.1, various cases were considered when the solution x^* of the equation

$$x=Ax \tag{6.15}$$

can be obtained as the limit of the successive approximations

$$x_n=Ax_{n-1}\qquad(n=1,2,\ldots). \tag{6.16}$$

In the cases considered in chapter 4, the initial approximation was selected in order that sequence (6.16) would be monotone. Then the conclusion about the convergence of this sequence can be made either from the complete continuity of the operator A (if the sequence is bounded with respect to the norm), or from the regularity of the cone (if the sequence is bounded in the sense of a partial ordering), or from the complete regularity of the cone (if the sequence is bounded with respect to the norm). The basic difficulty in the application of the theorems in chapter 4 is contained in the construction of the initial approximation.
In the case of equation (6.15) with a concave operator A, the last difficulty does not occur.

Theorem 6.6. *Let the equation* (6.15) *with a concave monotone operator* A *have a unique non-zero solution* x^* *in the cone* K.★ *Let either one of the following conditions be satisfied*:
6.6(a) *The cone* K *is regular, the operator* A *is continuous.*
6.6(b) *The cone* K *is normal, the operator* A *is completely continuous.*
Then the successive approximations (6.16) *converge with respect to the norm to* x^* *no matter what the initial approximation* $x_0 \in K$, $x_0 \neq \theta$ *is.*

Proof. We consider the conical segment $\langle v_0, w_0 \rangle$ where

$$v_0 = t_1 x^*, \quad w_0 = t_2 x^* \qquad (0 < t_1 < 1 < t_2). \tag{6.17}$$

It follows from the concavity of the operator A that

$$Av_0 = A(t_1 x^*) \geqslant t_1 Ax^* = v_0$$

and

$$Aw_0 = A(t_2 x^*) \leqslant t_2 Ax^* = w_0.$$

Thus, the operator A leaves every conical segment $\langle v_0, w_0 \rangle$ invariant.
It follows from 4.4 that the successive approximations (6.16) converge to x^* if the initial approximation belongs to one of the conical segments $\langle v_0, w_0 \rangle$. Moreover, it follows from Theorem 4.4 that the approximations (6.16) converge to x^* if one of the elements $A^p x_0$ belongs to some conical segment $\langle v_0\, w_0 \rangle$.
Let x_0 be an arbitrary non-zero element of K. It follows from condition (6.1) that

$$\alpha u_0 \leqslant Ax_0 \leqslant \beta u_0 \qquad (\alpha, \beta > 0)$$

★ It suffices that some power of the operator A be a u_0-concave operator

and

$$\alpha_0 u_0 \leqslant x^* = Ax^* \leqslant \beta_0 u_0 \qquad (\alpha_0, \beta_0 > 0),$$

where u_0 is the non-zero element of K appearing in the definition of the concavity of the operator.
Therefore

$$\frac{\alpha}{\beta_0} x^* \leqslant \alpha u_0 \leqslant Ax_0 \leqslant \beta u_0 \leqslant \frac{\beta}{\alpha_0} x^*,$$

that is, Ax_0 belongs to a conical segment $\langle v_0, w_0 \rangle$ where

$$v = \frac{\alpha}{\beta_0} x^*, \quad w_0 = \frac{\beta}{\alpha_0} x^*.$$

This completes the proof of the theorem.

6.1.7. *Convergence of successive approximations for u_0-concave operators*
Theorem 6.7. *Let the operator A be monotone and u_0-concave. Let the cone K be normal.*
Then the successive approximations (6.16) *converge with respect to the u_0-norm to a non-zero solution x^* of the equation* (6.15) *for an arbitrary non-zero initial approximation $x_0 \in K$ and if the cone K is normal, then convergence is with respect to the usual norm.*

Proof. It follows from condition (6.1) that

$$\alpha_0 u_0 \leqslant Ax^* = x^* \leqslant \beta_0 u_0 \qquad (\alpha_0, \beta_0 > 0).$$

Therefore (see page 21) the u_0-norm is equivalent to the x^*-norm. Thus, it suffices to prove that in the conditions of the theorem the approximations (6.16) converge to x^* with respect to the x^*-norm.
First let

$$v_n = Av_{n-1}, \quad v_0 = t_1 x^* \qquad (0 < t_1 < 1). \tag{6.18}$$

This sequence does not decrease where

$$t_1 x^* \leqslant v_n \leqslant x^* \qquad (n = 1, 2, \ldots).$$

We denote by ϱ_n $(n = 1, 2, \ldots)$ a maximal number for which the inequality

$$\varrho_n x^* \leqslant v_n \tag{6.19}$$

is satisfied. It is obvious that

$$0 < t_1 = \varrho_0 \leqslant \varrho_1 \leqslant \varrho_2 \leqslant \ldots \leqslant \varrho_n \leqslant \ldots \leqslant 1.$$

If we show that $\varrho_n \to 1$ then it will follow from this that the sequence (6.18) converges to x^* with respect to the x^*-norm.

We assume that

$$\lim_{n\to\infty} \rho_n = \gamma < 1 . \tag{6.20}$$

From the u_0-concavity of the operator A follows the existence of a number $\eta > 0$ such that

$$A(\gamma x^*) \geqslant (1+\eta)\gamma A x^* = (1+\eta)\gamma x^* ,$$

from which

$$A(tx^*) = A\left(\frac{t}{\gamma}\gamma x^*\right) \geqslant \frac{t}{\gamma} A(\gamma x^*) \geqslant (1+\eta)\, tx^* \qquad (0 \leqslant t \leqslant \gamma) .$$

In particular,

$$A(\varrho_n x^*) \geqslant (1+\eta)\varrho_n x^* \qquad (n = 0, 1, 2, \ldots) . \tag{6.21}$$

It follows from (6.19) and the monotoneity of the operator that

$$v_{n+1} = Av_n \geqslant A(\varrho_n x^*)$$

and by virtue of (6.21)

$$(1+\eta)\varrho_n x^* \leqslant v_{n+1} .$$

This means

$$(1+\eta)\varrho_n \leqslant \varrho_{n+1} \qquad (n = 0, 1, 2, \ldots) ,$$

from which it follows that

$$\varrho_n \geqslant (1+\eta)^n \varrho_0 \qquad (n = 1, 2, \ldots) ,$$

which contradicts (6.20).

Thus, $\varrho_n \to 1$.

We now consider the sequence

$$w_n = Aw_{n-1}, \quad w_0 = t_2 x^* \qquad (t_2 > 1) . \tag{6.22}$$

This sequence does not increase:

$$w_0 \geqslant w_1 \geqslant w_2 \ldots \geqslant w_n \geqslant \ldots \geqslant x^* .$$

We denote by ξ_n a minimal number for which the inequality

$$w_n \leqslant \xi_n x^* \qquad (n = 1, 2, \ldots) \tag{6.23}$$

is satisfied. It is obvious that

$$\xi_0 \geqslant \xi_1 \geqslant \xi_2 \geqslant \ldots \geqslant \xi_n \geqslant \ldots \geqslant 1 .$$

We shall show that

$$\lim_{n\to\infty} \xi_n = 1 .$$

Let us assume the contrary, i.e., that $\xi_n \to \gamma_0 > 1$. From the u_0-concavity of the operator A follows the existence of a number $\eta_0 > 0$ such that

$$Ax^* = A\left(\frac{1}{\gamma_0}\gamma_0 x^*\right) \geqslant \frac{1+\eta_0}{\gamma_0} A(\gamma_0 x^*) ,$$

that is,

$$A(\gamma_0 x^*) \leqslant \frac{\gamma_0}{1+\eta_0} Ax^* = \frac{\gamma_0 x^*}{1+\eta_0} ,$$

from which the inequalities

$$A(tx^*) = A\left(\frac{t}{\gamma_0}\gamma_0 x^*\right) \leqslant \frac{t}{\gamma_0} A(\gamma_0 x^*) \leqslant \frac{tx^*}{1+\eta_0} \qquad (t \geqslant \gamma_0)$$

follow. In particular,

$$A(\xi_n x^*) \leqslant \frac{\xi_n}{1+\eta_0} x^* \qquad (n = 0, 1, 2, \ldots)$$

and

$$w_{n+1} = Aw_n \leqslant A(\xi_n x^*) \leqslant \frac{\xi_n}{1+\eta_0} x^* ,$$

from which it follows that

$$\xi_{n+1} \leqslant \frac{\xi_n}{1+\eta_0} \qquad (n = 0, 1, 2, \ldots) .$$

This means that

$$\xi_n \leqslant \frac{\xi_0}{(1+\eta_0)^n} \qquad (n = 0, 1, 2, \ldots) ,$$

which contradicts the inequality $\xi_n \geqslant \gamma_0$.

We now consider the sequence (6.16) with an arbitrary non-zero initial approximation $x_0 \in K$. As in the proof of the preceding theorem, we convince ourselves that

$$v_0 = t_1 x^* \leqslant x_1 = Ax_0 \leqslant t_2 x^* = w_0$$

for some t_1 and t_2 which satisfy the inequalities

$$0 < t_1 < 1 < t_2 .$$

Then the successive approximations (6.25) satisfy the inequalities

$$v_n \leqslant x_{n+1} \leqslant w_n \qquad (n = 1, 2, \ldots) ,$$

where v_n and w_n are defined by the formulas (6.18) and (6.22). It follows from (6.19) and (6.23) that

$$\varrho_n x^* \leqslant x_{n+1} \leqslant \xi_n x^* \qquad (n=1, 2, \ldots) .$$

It follows from these inequalities that

$$\lim_{n\to\infty} \|x_n - x^*\|_{x^*} = 0 , \tag{6.24}$$

since $\varrho_n \to 1$ and $\xi_n \to 1$.
This completes the proof of the theorem.

The statement of the theorem remains valid if there is a u_0-concave operator A^p where p is some natural number. For the proof it suffices to remark that from Theorem 6.7 follows the convergence with respect to the u_0-norm of the sequences

$$A^k x_0, \quad A^{k+p} x_0, \quad A^{k+2p} x_0, \ldots \qquad (k=0, 1, \ldots, p-1)$$

to one and the same limit x^*. Therefore, sequence (6.16) converges to x^* with respect to the u_0-norm.
It was not assumed in the conditions of Theorem 6.7 that the operator A is continuous.
The conclusion concerning the convergence of the successive approximations with respect to the u_0-norm can also be made under the conditions of Theorem 6.6 if some power A^p of the operator A transforms every sequence which converges with respect to the usual norm into a sequence which converges with respect to the u_0-norm.

6.2. Existence of a continuum of characteristic vectors

6.2.1. *Completely continuous concave operators*

We first assume that the concave operator A is completely continuous and monotone.
Let an arbitrary number $\varepsilon > 0$ be given. We denote by γ a positive number such that from $\|x\| \leqslant r$ and $x \geqslant tu_0$ it follows that $t \leqslant \gamma$. Let

$$A(\gamma u_0) \geqslant \alpha_0 u_0 \qquad (\alpha_0 > 0) .$$

It then follows from condition (6.2) that for $0 \leqslant t \leqslant \gamma$

$$A(tu_0) \geqslant \frac{t}{\gamma} A(\gamma u_0) \geqslant \frac{\alpha_0}{\gamma} tu_0 .$$

The inequality obtained can be considered as condition (5.46) of Theorem 5.8. It follows from this theorem that A has a continuum of characteristic vectors and that these characteristic vectors form a continuous branch of infinite length.

We further assume that the cone K is normal. It then follows from Theorem 6.2 that the positive spectrum of the operator A forms an interval.

Let the uniqueness theorem be valid for the operator A (for example, A is u_0-concave, u_0-monotone or satisfies the conditions of Theorem 6.5). Then the positive spectrum forms an interval. In fact, if there is a largest number $\lambda_0>0$ in the positive spectrum to which the characteristic vector $x_0(Ax_0=\lambda x_0)$ corresponds, then all the remaining characteristic vectors $x(\lambda)$ satisfy the inequality $x(\lambda)\geqslant x_0$ by virtue of Theorem 6.1. It then follows from the semi-monotoneity of the norm that $\|x(\lambda)\|\geqslant a\|x_0\|$, that is, the operator A does not have characteristic vectors of a small norm, and this contradicts Theorem 5.8. Analogously, if in the positive spectrum there is a smallest number, then it is positive by virtue of condition (6.1). From Theorem 6.1 and the semi-monotoneity of the norm, it is easy to form the conclusion concerning the absence for the operator A of characteristic vectors which have a large norm. Again this contradicts Theorem 5.8.*

Let the positive spectrum form the interval $(\lambda_\infty, \lambda_\theta)$. The characteristic vectors $x(\lambda)$

$$Ax(\lambda)=\lambda x(\lambda) \qquad (\lambda_\infty<\lambda<\lambda_\theta)$$

form, by virtue of Theorem 6.1, a monotone function defined on $(\lambda_\infty, \lambda_\theta)$. This function $x(\lambda)$ is continuous since for an arbitrary $\lambda_0\in(\lambda_\infty, \lambda_\theta)$ and an arbitrary non-decreasing sequence $\lambda_n\to\lambda_0$ the characteristic vectors $x(\lambda_n)$ form a non-increasing compact sequence bounded below by the element $x(\lambda_0)$. The limit of the sequence $x(\lambda_n)$ will be a solution of the equation $Ax=\lambda_0 x$ and will, by virtue of the uniqueness theorem, coincide with $x(\lambda_0)$. Analogously, if $\lambda_n\to\lambda_0$ and λ_n does not increase, then the sequence $x(\lambda_n)$ does not decrease; its limit is a solution of the equation $Ax=\lambda_0 x$ and, by virtue of uniqueness, coincides with $x(\lambda_0)$.

We now combine the statements obtained above.

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* The fact that in a positive spectrum there is no greatest number λ_0 is easily proved even without the assumption of the normality of the cone since it follows from $x(\lambda)\geqslant x_0$, where $x_0\in K$ and $x_0\neq\theta$, that $\inf_\lambda\|x(\lambda)\|>0$.

Theorem 6.8. *The characteristic vectors of a concave and monotone completely continuous operator A form a continuous branch of infinite length. If the cone K is normal, then the positive spectrum forms an interval. Further, if the uniqueness theorem is valid for the operator A, then the positive spectrum is the interval $(\lambda_\infty, \lambda_\theta)$ where*

$$\lim_{\lambda \to \lambda_\theta} \|x(\lambda)\| = 0\,, \quad \lim_{\lambda \to \lambda_0} \|x(\lambda)\| = \infty\,,$$

where $Ax(\lambda) = \lambda x(\lambda)$. In this connection, the vector-function $x(\lambda)$ is single valued and decreases as λ increases.

6.2.2. *General case of a concave operator*

We now omit the assumption of complete continuity of the operator A. In this case, several statements of Theorem 6.6 are retained.

For the proof of the existence of a non-zero positive solution for the equation $Ax = \lambda x$, it is convenient to apply Theorem 4.1 which is relative to arbitrary monotone and not only concave operators. It is convenient to construct the conical segment $\langle v, w \rangle$ which is invariant for A, by means of methods studied in chapter 3.

In addition to general considerations, we note that for u_0-concave operators, from the inequalities

$$\lambda Av \geqslant v\,, \quad \lambda Aw \leqslant w \tag{6.25}$$

it automatically follows that

$$v \leqslant w\,. \tag{6.26}$$

Assuming the contrary, we denote by t_0 a maximal number such that $w \geqslant t_0 v$. Obviously, $t_0 < 1$. Let

$$\alpha_1 u_0 \leqslant Av \leqslant \beta_1 u_0\,, \quad \alpha_2 u_0 \leqslant Aw \leqslant \beta_2 u_0\,.$$

Then

$$w \geqslant \lambda Aw \geqslant \lambda \alpha_2 u_0 \geqslant \frac{\lambda \alpha_2}{\beta_1} Av \geqslant \frac{\alpha_2}{\beta_1} v\,,$$

from which it follows that $t_0 > 0$. It then follows from the u_0-concavity of the operator A that

$$w \geqslant \lambda Aw \geqslant \lambda A(t_0 v) \geqslant \lambda(1+\eta) t_0 Av \geqslant (1+\eta) t_0 v\,,$$

and this contradicts the maximality of t_0. Then the proof is retained if the operator A is u_0-monotone or satisfies the conditions of Theorem 6.5. Also, it is not difficult to see that for concave operators the inequality

(6.26) follows from (6.25) if either the first of the conditions (6.25) is replaced by a stronger condition $\lambda Av \geqslant v + \varepsilon u_0 (\varepsilon > 0)$ or the second of the conditions (6.25) is replaced by the condition $\lambda Aw + \varepsilon u_0 \leqslant w (\varepsilon > 0)$.

The existence of characteristic vectors becomes obvious for concave operators (as also for arbitrary monotone operators) if $A\theta \neq \theta$. In this case, we denote by β_0 a positive number such that $Au_0 \leqslant \beta v_0$. Then the operator $(1/\beta_0)A$ will leave the conical segment $\langle \theta, u_0 \rangle$ invariant, and the equation $Au_0 = \beta_0 u_0$ will have a non-zero positive solution if the cone is strongly minihedral or the cone is regular and A is continuous.

We have noted above that the uniqueness of the solution for the equation $x = A^p x$ follows from the uniqueness of the solution for the equation $x = Ax$ The analogous statement is valid for the problem concerning the existence of the solution. Let the equation $x = A^p x$ have, in some domain T, invariant for the operator A, a unique solution x^*. Then $Ax^* = A^p Ax^*$ and it follows from the uniqueness that $Ax^* = x^*$, that is, x^* is the solution of the equation $x = Ax$.

It follows from this consideration that all the statements of Theorem 6.8 remain valid if some power A^p of the operator A is a completely continuous u_0-concave operator.

The statements cited below in this subsection are valid for the operators some power of which is u_0-concave. We restrict ourselves to the case when the operator A itself is a u_0-concave operator.

Lemma 6.1. *Let the u_0-concave operator A be continuous and let the cone K be regular.**

Then the positive spectrum of the operator A contains, together with every point λ_0, completely some neighborhood of it.

Proof. Let $Ax_0 = \lambda_0 x_0 (x_0 \neq \theta)$ and

$$A(\tfrac{1}{2}x_0) \geqslant \frac{1+\eta}{2} Ax_0 = \frac{1+\eta}{2} \lambda_0 x_0 \qquad (\eta > 0).$$

Then for $\lambda_0 < \lambda < (1+\eta)\lambda_0$

$$\frac{1}{\lambda} A(\tfrac{1}{2}x_0) \geqslant \frac{1}{(1+\eta)\lambda_0} A(\tfrac{1}{2}x_0) \geqslant \tfrac{1}{2}x_0,$$

$$\tfrac{1}{2}Ax_0 \leqslant \frac{1}{\lambda_0} Ax_0 = x_0.$$

* If the cone K is strongly minihedral, then the continuity of the operator A is not necessary.

This means that the operators $(1/\lambda)A$ leave the conical segment $\langle \frac{1}{2}x_0, x_0\rangle$ invariant for $\lambda_0 < \lambda < (1+\eta)\lambda_0$ and, by virtue of Theorem 4.1, have for these values λ fixed points on this segment.
Let

$$Ax_0 = A(\tfrac{1}{2}\cdot 2x_0) \geqslant \frac{1+\eta_1}{2} A(2x_0) \qquad (\eta_1 > 0\,.$$

Then for $\lambda_0/(1+\eta_1) < \lambda < \lambda_0$, we have

$$\tfrac{1}{2}Ax_0 \geqslant \frac{1}{\lambda_0} Ax_0 = x_0\,,$$

$$\frac{1}{\lambda} A(2x_0) \leqslant \frac{2}{(1+\eta_1)\lambda} Ax_0 \leqslant \frac{2}{\lambda_0} Ax_0 = 2x_0\,,$$

that is, the operator $\frac{1}{2}Ax$ leaves the conical segment $\langle x_0, 2x_0\rangle$ invariant and has a fixed point on it by virtue of Theorem 4.1.
This completes the proof of the lemma.

It follows from this lemma and Theorem 6.2 that the positive spectrum of a continuous u_0-concave operator forms the interval (Λ_1, Λ_2) if K is regular. The vector-function $x(\lambda)$ whose values are characteristic vectors, is single-valued by virtue of Theorem 6.4, monotone by virtue of Theorem 6.1, and continuous. The continuity is proved literally as in the case of a completely continuous operator.
The statement of Lemma 6.1 remains valid if the cone K has only the property of normality and the u_0-concave operator A has in addition the property of complete continuity. The proof does not change.
We show in conclusion that

$$\lim_{\lambda\to\Lambda_1} \|x(\lambda)\| = \infty, \quad \lim_{\lambda\to\Lambda_2} \|x(\lambda)\| = 0\,, \tag{6.27}$$

if the cone is fully regular.
We assume, for example, that the first of the equalities (6.27) is not satisfied. Then the inequality

$$\sup_{\Lambda_1<\lambda<\Lambda_2} \|x(\lambda)\| < \infty \tag{6.28}$$

occurs. In fact, if for some sequence $\lambda_n \in (\Lambda_1, \Lambda_2)$

$$\lim_{n\to\infty} \|x(\lambda_n)\| = \infty\,, \tag{6.29}$$

then, without loss of generality, it can be assumed that

$$\|x(\lambda_{n+1})\| \geqslant m\|x(\lambda_n)\| \qquad (n=1, 2, \ldots) \tag{6.30}$$

(otherwise, we would take a subsequence) where m is a number such that from $\theta \leqslant x \leqslant y$ follows the inequality $\|x\| \leqslant m\|y\|$. The existence of the number m follows from Theorems 1.6 and 1.2: by virtue of the first theorem a regular cone is normal, and by virtue of the second, the norm is semi-monotone if the cone is normal. It follows from (6.30) and Theorem 6.1 that the sequence λ_n does not increase. Let $\lambda_n \to \lambda_0$. The equality $\lambda_0 = \Lambda_1$ occurs since otherwise $\lambda_0 > \Lambda_1$ and $x(\lambda_n)$ converges with respect to the norm to $x(\lambda_0)$ which contradicts (6.29). A subsequence λ_{n_k} (here some indices can be repeated) can be shown for an arbitrary sequence $\mu_k \in (\Lambda_1, \Lambda_2)$ converging to Λ_1 such that

$$\Lambda_1 < \mu_k < \lambda_{n_k} \qquad (k=1, 2, \ldots).$$

It then follows from Theorem 6.1 that

$$x(\mu_k) \geqslant x(\lambda_{n_k}) \qquad (k=1, 2, \ldots);$$

it follows from the semi-monotoneity of the norm that

$$\|x(\mu_k)\| \geqslant \frac{1}{m}\|x(\lambda_{n_k})\| \qquad (k=1, 2, \ldots),$$

and from these inequalities and from (6.29) it follows that the first of the equalities (6.27) is satisfied.

Thus, if the first of the equalities (6.27) is not satisfied, then the inequality (6.28) is satisfied.

It follows from (6.28) and from the full regularity of the cone K that $x(\lambda)$ as $\lambda \to \Lambda_1$ converges to some element x^*. Passing in the equality

$$Ax(\lambda) = \lambda x(\lambda)$$

to the limit as $\lambda \to \Lambda_1$, we obtain $Ax^* = \Lambda_1 x^*$ where $x^* \neq \theta$, since $x^* \geqslant x(\lambda)$ for $\lambda \in (\Lambda_1, \Lambda_2)$. Thus, Λ_1 is not a bound for the positive spectrum. We have arrived at a contradiction. The first of the equalities (6.27) is proved. The second equality (6.27) is proved analogously.

We formulate the results obtained in the form of a single theorem.

Theorem 6.9. *Let the cone K be fully regular. Let the continuous u_0-concave operator A have one positive characteristic vector.*

Then A has a continuum of characteristic vectors $x(\lambda)$ which are the values of single-valued continuous and decreasing vector-function, defined on a positive spectrum which forms some interval. The equalities (6.27) *are satisfied.*

It seems probable to us that the existence of characteristic vectors follows

from the continuity of a concave operator in a space with a fully regular cone (and perpaps, also, with fewer restrictions on the cone). However, this fact is not proved.

6.2.3. *Upper bound of the positive spectrum of a concave operator*
We continue the study of the spectrum of a positive and continuous u_0-concave operator A. We shall assume that the characteristic vectors $x(\lambda)$,

$$Ax(\lambda)=\lambda x(\lambda)\,, \tag{6.31}$$

form on some interval (Λ_1, Λ_2) a single-valued continuous and decreasing vector-function which satisfies the equalities (6.27). For this, it suffices that the conditions of Theorem 6.8 or Theorem 6.7 be satisfied.
Let $\|x(\lambda)\|\to 0$, then $\lambda\to\Lambda_2$. If $A\theta\neq\theta$, then it follows from (6.31) that $\Lambda_2=\infty$.
Let $A\theta=\theta$; then Λ_2 can also be equal to $+\infty$ and can be a finite positive number. We consider this last case.

Lemma 6.2. *Let the operator $A(A\theta=\theta)$ have a weak derivative $A'(\theta)$ with respect to a cone. Then $A'(\theta)$ is a majorant of the concave operator A.*

Proof. We assume that $A'(\theta)$ is not a majorant. Then a non-zero element $x\in K$ can be found such that $A'(\theta)x\geqq Ax$, that is, $A'(\theta)x-Ax\in K$.
We construct a linear positive functional $f(x)$ such that

$$f[A'(\theta)x-Ax]=a<0\,.$$

It is obvious that for $0<t<1$,

$$f\left[\frac{A'(\theta)tx-A(tx)}{t}\right]=a+f\left[\frac{tAx-A(tx)}{t}\right]\leqslant a\,.$$

Letting t tend to zero and noting that

$$\lim_{t\to 0} f\left[\frac{A'(\theta)tx-A(tx)}{t}\right]=0\,,$$

we arrive at the inequality $a\geqslant 0$. We have arrived at a contradiction. This completes the proof of the lemma.

In the conditions of Lemma 6.2, $\Lambda_2<\infty$. In fact, for $\lambda\in(\Lambda_1,\Lambda_2)$,

$$\lambda=\frac{\|Ax(\lambda)\|}{\|x(\lambda)\|}\leqslant\left\|\frac{Ax(\lambda)-A'(\theta)x(\lambda)}{\|x(\lambda)\|}\right\|+\|A'(\theta)\|\,,$$

and the first term in the right member is bounded for small $\|x(\lambda)\|$ since the ratio

$$\frac{Ax(\lambda)-A'(\theta)x(\lambda)}{\|x(\lambda)\|}$$

converges weakly to zero.
It is assumed below that $A'(\theta)$ is a strong Fréchet derivative with respect to a cone.

Theorem 6.10. *Let $A'(\theta)$ be completely continuous. Then the upper bound Λ_2 of the positive spectrum of the operator A is a positive characteristic value of the operator $A'(\theta)$.*

Proof. We remark first that, by virtue of Lemma 6.2, the operator A is a monotone minorant of the operator $A'(\theta)$. It then follows from Theorem 5.8 that $A'(\theta)$ has positive characteristic vectors. Consequently, $A'(\theta)$ has positive characteristic values.
We choose a sequence $\lambda_n \to \Lambda_2$, such that the elements $A'(\theta)[x(\lambda_n)/\|x(\lambda_n)\|]$ converge to some element $y_0 \in K$. Passing in the equalities

$$\frac{\lambda_n x(\lambda_n)}{\|x(\lambda_n)\|} = \frac{Ax(\lambda_n)-A'(\theta)x(\lambda_n)}{\|x(\lambda_n)\|} + A'(\theta)\frac{x(\lambda_n)}{\|x(\lambda_n)\|} \tag{6.32}$$

to the limit, we convince ourselves of the convergence of the sequence $x(\lambda_n)/\|x(\lambda_n)\|$ to the element $x_0=(1/\Lambda_2)y_0 \in K$. After the passage to the limit, we obtain the equality

$$A'(\theta)x_0=\Lambda_2 x_0\,. \tag{6.33}$$

This completes the proof of the theorem.

If $A'(\theta)$ has a single positive characteristic value, then Theorem 6.10 allows the determination of an upper bound of the positive spectrum.

6.2.4. *Lower bound of the positive spectrum*
The lower bound Λ_1 of the positive spectrum can be equal to zero. In the case of $\Lambda_1>0$ the investigation is developed by the same method which was applied in the preceding subsection for the case when $\Lambda_2<\infty$.
Analogously to Lemma 6.2, it is proved that the weak derivative $A'(\infty)$ with respect to the cone K (if this derivative exists) is a minorant of the operator A on K. In this connection, if the operator $A'(\infty)$ is a u_0-positive operator, then it follows from Theorem 5.10 that all the characteristic

values of the operator A are no less (and consequently, greater) than the positive characteristic value of the operator $A'(\infty)$. Therefore, from arguments which are analogous to the proof of Theorem 6.10, follows

Theorem 6.11. *Let the u_0-concave positive and continuous operator A have a strong asymptotic derivative $A'(\theta)$ with respect to a cone. Let the operator $A'(\theta)$ be completely continuous and u_0-positive.*
Then λ_1 is a positive characteristic value of the operator $A'(\infty)$.

If $A'(\infty)$ does not have the property of u_0-positiveness, then λ_1 will either be a positive characteristic value of the operator $A'(\infty)$ or be equal to zero. If $A'(\infty)$ does not have positive characteristic values, then $\lambda_1 = 0$.
We note that Theorems 6.10 and 6.11 remain valid for some cases when $A'(\theta)$ and $A'(\infty)$ are not completely continuous.

6.2.5. *Operators which are concave on part of the cone*
Let two cones K and K_0 be given in the space E. As usual, $K_0 \subset K$ and a partial ordering in E is introduced by means of the larger cone K.
We assume that the operator A is given on K_0 and leaves K_0 invariant. Let $u_0 \in K_0$.
All the definitions given in the present chapter carry over, without change, to operators which leave the cone K_0 invariant; it is necessary only to require that the corresponding inequalities are satisfied only for elements of K_0. For example, the operator A is *u_0-concave* on K_0, if for an arbitrary non-zero $x \in K_0$ the inequalities (6.1) are valid and if for every $x \in K_0$ such that $\alpha_1(x)u_0 \leqslant x \leqslant \beta_1(x)u_0$ ($\alpha_1(x) > 0$, $\beta_1(x) > 0$), and every positive number $t_0 < 1$ there can be found a number $\eta = \eta(x, t_0) > 0$ such that

$$A(t_0 x) \geqslant (1+\eta) t_0 A x .$$

All the theorems proved up till now in this chapter remain valid if we turn from operators which are concave on the entire cone K to operators which are concave on K_0. The proofs in this connection do not change.
The proofs of Theorems 6.1, 6.3, 6.4 also do not change in the case when the operator is concave on some set $T_0 \subset K$ of a more complicated nature than the cone K_0. It is necessary to require that the set T_0 contain, together with every element x, all the elements tx ($0 \leqslant t \leqslant 1$). Examples of such sets can be the conical segments $\langle \theta, w_0 \rangle$, the intersections K_r of the cone K with the sphere $\|x\| \leqslant r$, the intersections $K_0 \langle \theta, w_0 \rangle$ of the cone K_0 with the conical segment $\langle \theta, w_0 \rangle$, and so on.

Theorem 6.2 remains valid in the transition to the set T_0, if it, together with every element x_0, contains the conical segment $\langle\theta, x_0\rangle$. If A_0 leaves the cone $K_0 \subset K$ invariant, then it suffices that T_0 contains all the sets $K_0\langle\theta, x_0\rangle (x_0 \in T_0)$.

Theorem 6.5 remains valid if $AT \subset T$.

The theorems concerning the convergence of the successive approximations require a modification in their formulation. We denote only that the successive approximations

$$x_n = Ax_{n-1} \qquad (n = 1, 2, \ldots)$$

converge to a unique non-zero solution x^* of the equation $x = Ax$ on $K_0\langle\theta, u_0\rangle$ for an arbitrary non-zero initial approximation $x_0 \in K\langle\theta, u_0\rangle$, if the completely continuous operator A is concave on $K_0\langle\theta, u_0\rangle$ and if $Au_0 \leqslant u_0$.

It is not difficult to generalize the theorems of the present section.

6.3. Equations with convex operators

6.3.1. *Convex operators*

Let the set $T \subset K$ be closed and have the property that together with every element x it contains all the points tx where $0 \leqslant t \leqslant 1$. Let u_0 be a fixed non-zero element of T.

We call the operator A *u_0-convex* on T if for every non-zero $x \in T$ the inequalities

$$\alpha u_0 \leqslant Ax \leqslant \beta u_0 , \tag{6.34}$$

where $\alpha = \alpha(x)$, $\beta = \beta(x)$ are positive, are satisfied, and if for every $x \in T$ such that $\alpha_1(x)u_0 \leqslant x \leqslant \beta_1(x)u_0$ $(\alpha_1(x), \beta_1(x) > 0)$ and for every positive number $t_0 \in (0, 1)$ there can be found an

$$\eta = \eta(x, t_0) > 0$$

such that

$$A(t_0 x) \leqslant (1 - \eta) t_0 Ax . \tag{6.35}$$

The fundamental theorems proved in the preceding two sections for equations with concave operators do not carry over to equations with convex operators. In particular, the uniqueness theorem of a positive solution is not true.

As an example, we can consider in a two-dimensional space of points

$x=\{\xi_1, \xi_2\}$ the operator

$$A\{\xi_1, \xi_2\} = \left\{\xi_1^2+\xi_2, \frac{1}{\varepsilon}\xi_2^2+\varepsilon^2\xi_1\right\}, \tag{6.36}$$

where ε is a sufficiently small positive number. Let the cone K consist of elements with non-negative coordinates $u_0=\{1, 1\}$. The u_0-convexity of operator (6.36) is obvious. At the same time, the equation $x=Ax$ has three solutions in the cone. In order to find these solutions, it is necessary to find numbers ξ_1 and ξ_2 such that

$$\xi_1=\xi_1^2+\xi_2, \quad \xi_2 = \frac{1}{\varepsilon}\xi_2^2+\varepsilon^2\xi_1 . \tag{6.37}$$

The equalities (6.37) can be considered as equations of two parabolas in the $\{\xi_1, \xi_2\}$ plane. The graphs of these parabolas, as is easily seen, have

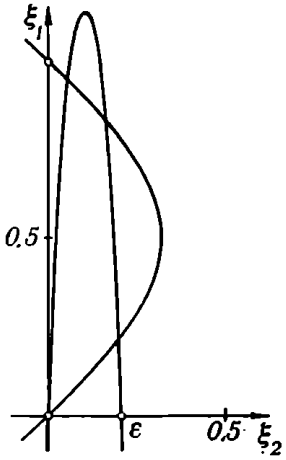


Fig. 1

the form which is shown in Fig. 1. Therefore, three solutions exist in the cone K.

At the same time, there exist convex operators on the entire cone K for which the uniqueness of solutions is clear from other considerations. An example of a similar situation is given below in chapter 7.

The determination of the fewest number of restrictions for which the equations with connex operators have a uniqne solution is an important unsolved problem.

6.3.2. *Uniqueness principle*

Lemma 6.3. *Let the equation $x=Ax$ with the monotone u_0-convex operator A have two non-zero solutions x^* and x^{**}. Then*

$$x^* \overline{\leqslant} tx^{**} \qquad (0<t<1)\,. \tag{6.38}$$

Proof. We assume that the inequality $x^* \leqslant t^* x^*$ is satisfied for some $t^* \in (0, 1)$. We denote by t_0 a number such that $x^* \leqslant t_0 x^{**}$ and $x^* \overline{\leqslant} tx^{**}$ for $t<t_0$. It follows from (6.34) that $t_0>0$.
By virtue of the monotoneity of A and by (6.35)

$$x^* = Ax^* \leqslant A(t_0 x^{**}) \leqslant (1-\eta)\, t_0 Ax^{**} = (1-\eta)\, t_0 x^{**}\,,$$

which contradicts the minimality of t_0.
This completes the proof of the lemma.

If the u_0-convex operator A has the additional property that the inequality $Ay \geqslant Ax + \varepsilon_0 u_0$, where $\varepsilon_0>0$ follows from $\theta \leqslant x \leqslant y$ and $x \neq y$, then the relation $x^* \overline{\leqslant} x^{**}$ which is stronger than (6.38) holds. The proof is virtually unchanged.

In the following section, Lemma 6.3 will be applied in the form of the following uniqueness principle: *if in the conditions of Lemma* 6.3 *the cone K is solid and one of the elements $x^* - x^{**}$, $x^{**} - x^*$ can be either zero or an interior element of the cone, then*

$$x^* = x^{**}\,.$$

6.4. Applications to the problem about the points of bifurcation

6.4.1. *Formulation of the problem*

We consider the equation

$$Ax = \lambda x \tag{6.39}$$

with a numerical parameter which for simplicity we shall assume to be positive. Let $A\theta = \theta$; then the equation (6.39) has a zero solution θ for all values of the parameter λ.
The number λ_0 is called a *point of bifurcation* for the operator A if to every $\varepsilon>0$ there corresponds a value which satisfies the inequality

$$|\lambda - \lambda_0| < \varepsilon\,,$$

for which the equation (6.39) has at least one non-zero solution satisfying

the condition

$$\|x(\lambda)\| < \varepsilon .$$

Everywhere below it is assumed that the operator A has a strong Fréchet derivative at the point θ:

$$A'(\theta) = B .$$

Lemma 6.4. *If the operator $B - \lambda_0 I$ has a continuous inverse, then λ_0 is not a point of bifurcation.*

Proof. Since $B - \lambda_0 I$ has a continuous inverse operator, then

$$\|Bx - \lambda_0 x\| \geqslant k_0 \|x\| \qquad (x \in E) ,$$

where $k_0 > 0$. We choose a number r_0 such that for $\|x\| \leqslant r_0$ the inequality

$$\|Ax - Bx\| \leqslant \frac{k_0}{3} \|x\|$$

is satisfied. Then for $|\lambda - \lambda_0| < k_0/3$, $\|x\| \leqslant r_0$,

$$\|Ax - \lambda x\| \geqslant \|Bx - \lambda_0 x\| - \|Ax - Bx\| - |\lambda_0 - \lambda| \, \|x\| \geqslant \frac{k_0}{3} \|x\| .$$

This means for $|\lambda - \lambda_0| < k_0/3$ the equation (6.39) does not have non-zero solutions which satisfy the inequality $\|x\| < r_0$.
This completes the proof of the lemma.

Thus, it is necessary to look for the points of bifurcation among the points of the spectrum of the operator B.
We assume that the positive number λ_0 is an isolated point of the spectrum of the linear operator B. It turns out (see M. A. Krasnosel'skii [5]) that this is not sufficient in order that λ_0 be a point of bifurcation (even if λ_0 has finite multiplicity!). Additional assumptions are required either about the multiplicity of the characteristic value λ_0 or about the properties of the derivatives of higher order of the operator A at the point θ.
If it is known that the characteristic value λ_0 of the linear operator A is a point of bifurcation, then new natural questions arise. The basic ones are relative to the properties of the collection of those values of λ which are near to λ_0 for which the equation (6.39) has non-zero solutions: those values λ are positioned with respect to both sides of λ_0 or to one of the values (but which one?); the values λ completely fill some interval one of whose ends is λ_0, and so on. Moreover, it is important to know how many small non-zero solutions for a given λ there are, and how these solutions change with the changes of λ.

6.4.2. *Basic result*

We restrict ourselves to the consideration of the case when $\lambda_0 > 0$ and λ_0 is a simple characteristic value of the linear operator $\boldsymbol{B}$. We assume that every element $x \in E$ allows the unique representation in the form

$$x = \boldsymbol{P}x + \xi(x)h_0 , \tag{6.40}$$

where h_0 is a normalized ($\|h\| = 1$) characteristic vector of the operator $\boldsymbol{B}$ corresponding to the characteristic value λ_0,

$$\boldsymbol{B}h_0 = \lambda_0 h_0 , \tag{6.41}$$

$\boldsymbol{P}$ is the linear operator of the projection on the subspace E_0 which is invariant for the operator $\boldsymbol{B}$, and $\xi(x)$ is a linear functional satisfying the conditions

$$\xi(h_0) = 1 , \quad \xi(x) = 0 \qquad (x \in E_0) . \tag{6.42}$$

If $\boldsymbol{B}$ is considered only on E_0, then λ_0 will not belong to its spectrum

$$(\boldsymbol{B} - \lambda_0 I)E_0 = E_0 , \quad \|\boldsymbol{B}x - \lambda_0 x\| \geqslant \frac{1}{m_0}\|x\| \qquad (x \in E_0) . \tag{6.43}$$

If $\boldsymbol{B}$ is a self-conjugate operator, then the formula (6.40) attains a particularly simple nature: $\boldsymbol{P}$ is the operator of the orthogonal projection on the orthogonal subspace E_0, $\xi(x) = (x, h_0)$ to h_0.

The operator $\boldsymbol{B}$ evidently commutes with $\boldsymbol{P}$ and the formula

$$\boldsymbol{B}x = \boldsymbol{P}\boldsymbol{B}x + \lambda_0 \xi(x)h_0 \qquad (x \in E) \tag{6.44}$$

holds.

We shall assume that the operator $A(A\theta = \theta)$ allows the representation

$$Ax = \boldsymbol{B}x + \boldsymbol{C}x + \boldsymbol{D}x , \tag{6.45}$$

where $\boldsymbol{B}$ is the linear operator described above, $\boldsymbol{C}$ is a homogeneous operator of some order $s > 1$, that is,

$$\boldsymbol{C}(tx) \equiv t^s \boldsymbol{C}x \qquad (-\infty < t < \infty, \ x \in E) \tag{6.46}$$

and

$$\|\boldsymbol{C}x_1 - \boldsymbol{C}x_2\| \leqslant q_0 r^{s-1}\|x_1 - x_2\| \qquad (\|x_1\|, \ \|x_2\| \leqslant r) , \tag{6.47}$$

$\boldsymbol{D}$ is an operator of a higher order of smallness than s, that is,

$$\|\boldsymbol{D}x\| = o(\|x\|^s) , \tag{6.48}$$

and

$$\|\boldsymbol{D}x_1 - \boldsymbol{D}x_2\| \leqslant q_1(r)\|x_1 - x_2\| \qquad (\|x_1\|, \ \|x_2\| \leqslant r) , \tag{6.49}$$

where

$$\lim_{r\to 0}\frac{q_1(r)}{r^{s-1}}=0\,. \tag{6.50}$$

It will be shown below that for the enumerated conditions, λ_0 will be a point of bifurcation for the operator A. In order to obtain an answer to the question for what values λ near to λ_0 the equation (6.39) has solutions with small norms (and how many solutions), it is natural to apply the following approximate calculation plan.

We shall look for approximate non-zero solutions $\tilde{x}(\lambda)$ of the equation (6.39) in the form

$$\tilde{x}(\lambda)=\alpha h_0\,. \tag{6.51}$$

We remark that the operator A is approximately equal to the operator $\tilde{A}=B+C$. For the definition of the values α we make use of the equality

$$\xi(\tilde{A}\tilde{x}-\lambda\tilde{x})=0\,. \tag{6.52}$$

The equation (6.52) can be written in the form

$$\alpha^s\xi(Ch_0)+(\lambda_0-\lambda)\alpha=0\,.$$

The zero solution of this last equation corresponds to the trivial solution θ of the equation (6.39). The non-zero solutions are defined from the equation

$$\alpha^{s-1}\xi(Ch_0)=\lambda-\lambda_0\,. \tag{6.53}$$

It is clear that this approximate equation can be used only in the case when

$$\xi(Ch_0)\neq 0\,. \tag{6.54}$$

This assumption is described as the "general case."

It turns out that the conclusions which can be obtained upon the analysis of the approximate equation (6.53) accurately correspond to the questions formulated above.

The following theorem holds.

Theorem 6.12. *Let the condition* (6.54) *be satisfied. Then numbers* $r_0>0$ *and* $\delta_0>0$ *can be found such that the following statements are valid*:

(1) *Equation* (6.39) *does not have non-zero solutions for* $\lambda=\lambda_0$ *in the sphere* $\|x\|\leqslant r_0$.

(2) *If* s *is even, then for* $|\lambda-\lambda_0|<\delta_0(\lambda\neq\lambda_0)$ *the equation* (6.39) *has in the sphere* $\|x\|\leqslant r_0$ *a unique continuous non-zero solution* $x(\lambda)$, *depending on*

λ, *where*

$$\operatorname{sign} \xi[x(\lambda)] = \operatorname{sign} \xi(Ch_0) \cdot \operatorname{sign}(\lambda - \lambda_0),$$

$$\|x(\lambda) - \xi[x(\lambda)]h_0\| \leqslant M\,|\xi[x(\lambda)]|^s \tag{6.55}$$

and

$$M_1(\lambda)|\lambda - \lambda_0|^{1/(s-1)} \leqslant \|x(\lambda)\| \leqslant M_2(\lambda)|\lambda - \lambda_0|^{1/(s1-)}, \tag{6.56}$$

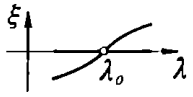
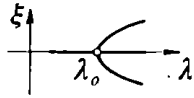
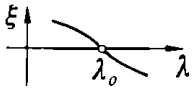
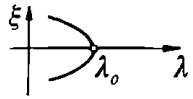
where

$$\lim_{\lambda \to \lambda_0} M_1(\lambda) = \lim_{\lambda \to \lambda_0} M_2(\lambda) = |\xi(Ch_0)|^{1/(s-1)}.$$

(3) *If s is odd and $\xi(Ch_0) > 0$, then the equation* (6.39) *does not have in the sphere $\|x\| \leqslant r_0$ non-zero solutions for $\lambda \in (\lambda_0 - \delta_0, \lambda)$ and has exactly two continuous non-zero solutions $x_1(\lambda)$ and $x_2(\lambda)$ depending on λ for $\lambda \in (\lambda_0, \lambda_0 + \delta_0)$. These solutions satisfy the conditions* (6.55), (6.56) *and*

$$\operatorname{sign} \xi[x_1(\lambda)] > 0, \quad \operatorname{sign} \xi[x_2(\lambda)] < 0. \tag{6.57}$$

(4) *If s is odd and $\xi(Ch_0) < 0$, then the equation* (6.39) *does not have in the sphere $\|x\| \leqslant r_0$ non-zero solutions for $\lambda \in (\lambda_0, \lambda_0 + \delta_0)$ and has exactly two non-zero solutions $x_1(\lambda)$ and $x_2(\lambda)$ continuously depending on λ for $\lambda \in (\lambda_0, \delta_0 + \lambda_0)$. These solutions satisfy the conditions* (6.55), (6.56), *and* (6.57). Theorem 6.12 becomes apparent if illustrative graphs of the functions $\xi[x(\lambda)]$ are drawn for different cases. These graphs have the form:

	S EVEN	*S ODD*
$\xi(Ch_0) > 0$		
$\xi(Ch_0) < 0$		

In many problems of mechanics (origin of waves, loss of stability in elastic systems and so on), the parameter λ in the equation (6.39) has the form $\lambda = 1/\mu$ and non-zero solutions arise only for the values of the parameter μ greater than the bifurcational value $\mu_0 = 1/\lambda_0$ (μ_0 is the critical rate of movement of a liquid, μ_0 is the Euler critical load and so on). It follows from Theorem 6.12 that the operators A in the equation (6.39)

describing such processes must appear in the representation (6.45) in which, generally speaking, the terms C of odd order, where the sign of these odd terms is opposite to the sign of the linear term succeed the linear term B.

The present section is concerned with the proof of Theorem 6.12. The method of investigation applied essentially does not depend on the multiplicity of the characteristic value λ_0. However, the case of multiple characteristic value requires the introduction of concepts from combinatorial topology in connection with which we are not here concerned with this case.

6.4.3. *Curve of suspected points*

The operator $B-\lambda_0 I$ has, by assumption, the continuous inverse $R(\lambda_0)$ on the invariant subspace E_0. Let $\|R(\lambda_0)\| \leqslant m_0$.

As is known, the operators $B-\lambda I$ for values of λ near to λ_0 also have inverses $R(\lambda)$ on E_0. It is not difficult to see that $R(\lambda)$ is representable by the series

$$R(\lambda)x = R(\lambda_0)x + (\lambda_0-\lambda)R^2(\lambda_0)x + \ldots$$
$$\ldots + (\lambda_0-\lambda)^n R^{n+1}(\lambda_0)x + \ldots \qquad (x \in E_0),$$

from which

$$\|R(\lambda)\| \leqslant m_0 + |\lambda_0-\lambda| m_0^2 + \ldots + |\lambda_0-\lambda|^n m_0^{n+1} + \ldots,$$

that is

$$\|R(\lambda)\| \leqslant \frac{m_0}{1-|\lambda_0-\lambda| m_0}. \tag{6.58}$$

All additional arguments are developed from the assumption that only those values λ are considered for which

$$\|R(\lambda)\| \leqslant 2m_0. \tag{6.59}$$

By virtue of (6.58) the last inequalities are satisfied if

$$|\lambda-\lambda_0| \leqslant \frac{1}{2m_0}. \tag{6.60}$$

We fix λ and look for the elements x such that

$$P(Ax-\lambda x) = \theta. \tag{6.61}$$

In this connection, we shall be interested only in those solutions which have a sufficiently small norm.

It turns out that equation (6.61) has a continuum of solutions. In this

connection, a unique solution x of the equation (6.61) with a small norm corresponds to every sufficiently small number α such that $\xi(x)=\alpha$. We denote this solution by x_α. Then

$$x_\alpha = \boldsymbol{P}x_\alpha + \alpha h_0 \,. \tag{6.62}$$

In order to prove the existence of solutions of the form (6.62) of the equation (6.61) we make use of the representation (6.45) of the operator A and the equality (6.44). Then equation (6.61) takes the form

$$(\boldsymbol{B}-\lambda \boldsymbol{I})\boldsymbol{P}x + \boldsymbol{P}(\boldsymbol{C}x+\boldsymbol{D}x) = \theta \,.$$

If the solution of the last equation is looked for in the form (6.62), then this equation can be written in the form

$$(\boldsymbol{B}-\lambda \boldsymbol{I})\boldsymbol{P}x_\alpha + \boldsymbol{P}[\boldsymbol{C}(\boldsymbol{P}x_\alpha+\alpha h_0)+\boldsymbol{D}(\boldsymbol{P}x_\alpha+\alpha h_0)] = \theta$$

or

$$\boldsymbol{P}x_\alpha = -\boldsymbol{R}(\lambda)\boldsymbol{P}[\boldsymbol{C}(\boldsymbol{P}x_\alpha+\alpha h_0)+\boldsymbol{D}(\boldsymbol{P}x_\alpha+\alpha h_0)] \,. \tag{6.63}$$

We introduce for consideration the operators $\boldsymbol{S}(\alpha)$:

$$\boldsymbol{S}(\alpha)y = -\boldsymbol{R}(\lambda)\boldsymbol{P}[\boldsymbol{C}(y+\alpha h_0)+\boldsymbol{D}(y+\alpha h_0)] \qquad (y\in E) \,. \tag{6.64}$$

By virtue of conditions (6.47)–(6.50), a number r_0 can be chosen such that the operators $\boldsymbol{S}(\alpha)$ satisfy the conditions of the principle of contraction mappings for $\|\alpha\| \leqslant r_0$; in the sphere $T\{\|y\| \leqslant r_0,\ y\in E\}$ the operators $\boldsymbol{S}(\alpha)$ transform the sphere T into itself and on the sphere T satisfy the Lipschitz condition

$$\|\boldsymbol{S}(\alpha)y_1 - \boldsymbol{S}(\alpha)y_2\| \leqslant q\|y_1-y_2\| \qquad (y_1, y_2\in E_0,\ \|y_1\|,\ \|y_2\| \leqslant r_0) \tag{6.65}$$

with the constant $q<1$. From the principle of contraction mappings follows the existence and uniqueness in the sphere T of solutions of the equation

$$y = \boldsymbol{S}(\alpha)y \tag{6.66}$$

for $|\alpha| \leqslant r$. The equation (6.66) coincides with the equation (6.63).

The vector-function (6.62) is strongly continuous with respect to α and, moreover, satisfies the Lipschitz condition for α. In fact, from the equalities

$$\boldsymbol{P}x_{\alpha_1} = -\boldsymbol{R}(\lambda)\boldsymbol{P}[\boldsymbol{C}(\boldsymbol{P}x_{\alpha_1}+\alpha_1 h_0)+\boldsymbol{D}(\boldsymbol{P}x_{\alpha_1}+\alpha_1 h_0)] \,,$$

$$\boldsymbol{P}x_{\alpha_2} = -\boldsymbol{R}(\lambda)\boldsymbol{P}[\boldsymbol{C}(\boldsymbol{P}x_{\alpha_2}+\alpha_2 h_0)+\boldsymbol{D}(\boldsymbol{P}x_{\alpha_2}+\alpha_2 h_0)]$$

and from (6.65), it follows that

$$\|\boldsymbol{P}x_{\alpha_1} - \boldsymbol{P}x_{\alpha_2}\| \leqslant q(\|\boldsymbol{P}x_{\alpha_1} - \boldsymbol{P}x_{\alpha_2}\| + |\alpha_1-\alpha_2|) \,,$$

from which

$$\|Px_{\alpha_1}-Px_{\alpha_2}\| \leqslant \frac{q}{1-q}|\alpha_1-\alpha_2| . \tag{6.67}$$

Every solution of the equation $Ax=\lambda x$ is simultaneously a solution of the equation (6.61). Therefore non-zero solutions of the equation $Ax=\lambda x$ can be looked for and it is necessary to look only on the curve $x=x_\alpha$ which we shall denote by the letter Γ. This essentially simplifies the problem since the curve Γ is homeomorphic to a segment.

We note one property of the curve Γ.

By virtue of (6.46) and (6.48),

$$\|S(\alpha)\theta\| \leqslant \|R(\lambda)\| \, \|P\| \, [\|C(\alpha h_0)\|+D(\alpha h_0)] \leqslant L_1|\alpha|^s. \tag{6.68}$$

The inequality

$$\begin{aligned}\|Px_\alpha\| = \|S(\alpha)Px_\alpha\| &\leqslant \|S(\alpha)\theta\| + \|S(\alpha)Px_\alpha - S(\alpha)\theta\| \\ &\leqslant q\|Px_\alpha\| + L_1(\alpha)^s ,\end{aligned}$$

then follows from (6.65) from which

$$\|Px_\alpha\| \leqslant \frac{L_1}{1-q}|\alpha|^s \qquad (|\alpha| \leqslant r_0) . \tag{6.69}$$

It follows from the derivation of the inequality (6.65) that $q \to 0$ as $r_0 \to 0$.

6.4.4. *Existence of non-zero small solutions*

In order for the point x_α on the curve Γ to be a solution of the equation $Ax=\lambda x$, it is necessary and sufficient that the equality

$$\xi(Ax_\alpha - \lambda x_\alpha)=0 \tag{6.70}$$

be satisfied, which we rewrite in the form

$$\xi\{C(Px_\alpha+\alpha h_0)\}+\xi\{D(Px_\alpha+\alpha h_0)\}=(\lambda-\lambda_0)\alpha . \tag{6.71}$$

We introduce the notation

$$\varphi(\alpha)=\xi\{C(Px_\alpha+\alpha h_0)-C(\alpha h_0)+D(Px_\alpha+\alpha h_0)\} . \tag{6.72}$$

By virtue of the conditions (6.47)–(6.50) and the inequalities (6.69),

$$\varphi(\alpha)=\varphi_0(\alpha)|\alpha|^s\xi(Ch_0), \ \varphi_0(\alpha)=o(1) . \tag{6.73}$$

The equality (6.71) can be rewritten in the form

$$\xi(Ch_0)\alpha^s[1+\varphi_0(\alpha)]=\alpha(\lambda-\lambda_0) . \tag{6.74}$$

The existence or lack of non-zero solutions for the equation (6.74) is equivalent to the existence or lack of non-zero solutions for the equation $Ax=\lambda x$.

We choose the number $\alpha_0 \in (0, r_0)$ in order that for $|\alpha| \leqslant \alpha_0$ the inequality

$$|\varphi_0(\alpha)| < \tfrac{1}{2}$$

is satisfied. We shall look for non-zero solutions of the equation (6.74) which satisfy the condition $|\alpha| \leqslant \alpha_0$. These non-zero solutions are defined from the equation

$$[1+\varphi_0(\alpha)]\alpha^{s-1} = \frac{\lambda-\lambda_0}{\xi(Ch_0)}\,. \tag{6.75}$$

The analysis of the solutions is easily carried out from a graph.
The equation (6.75) does not have non-zero solutions for $\lambda = \lambda_0$. With this, the first statement of Theorem 6.12 is proved.

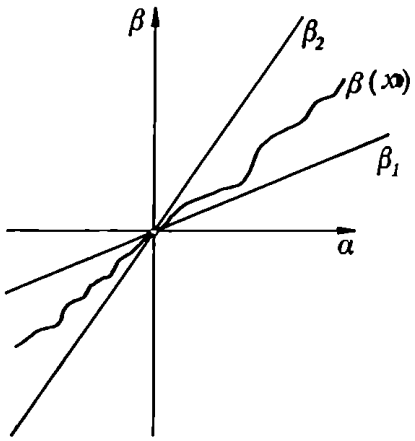


Fig. 2

The graph of the continuous function

$$\beta = [1+\varphi_0(\alpha)]\alpha^{s-1} \tag{6.76}$$

is contained between the graphs of the two functions

$$\beta_1 = \tfrac{1}{2}\alpha^{s-1}\,, \quad \beta_2 = \tfrac{3}{2}\alpha^{s-1}\,.$$

For even s (Fig. 2 is drawn for the case when $s=2$) the function β assumes positive values for $\alpha > 0$ and negative values for $\alpha < 0$. Therefore, the equation (6.75) has positive solutions and does not have negative solutions if

$$\operatorname{sign}(\lambda-\lambda_0) = \operatorname{sign}\xi(Ch_0)\,.$$

Analogously, the equation (6.75) has negative solutions and does not have positive solutions if

$$\operatorname{sign}(\lambda-\lambda_0) = -\operatorname{sign}\xi(Ch_0).$$

For odd s (Fig. 3 is drawn for the case when $s=3$)

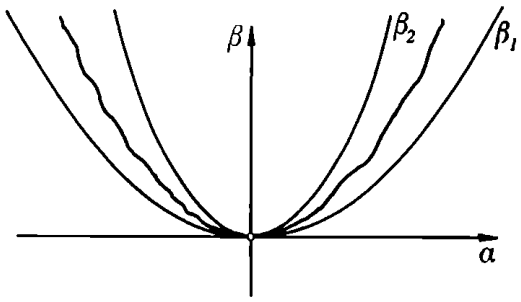


Fig. 3

the function (6.76) takes on only positive values. Therefore, the equation (6.75) does not have non-zero solutions if

$$\operatorname{sign}(\lambda-\lambda_0) = -\operatorname{sign}\xi(Ch_0),$$

and has at least one positive and at least one negative solution if

$$\operatorname{sign}(\lambda-\lambda_0) = \operatorname{sign}\xi(Ch_0).$$

The result obtained completely proves that part of the statement of Theorem 6.12 relative to the existence or absence of non-zero solutions.

6.4.5. *Estimation of the norms of non-zero solutions*

It follows from (6.75) that for the solutions $x(\lambda)$ of the equation $Ax=\lambda x$ (if these solutions exist) the inequalities

$$N_1(\lambda)|\lambda-\lambda_0|^{1/(s-1)} \leqslant |\xi[x(\lambda)]| \leqslant N_2(\lambda)|\lambda-\lambda_0|^{1/(s-1)} \tag{6.77}$$

hold, where

$$\lim_{\lambda\to\lambda_0} N_1(\lambda) = \lim_{\lambda\to\lambda_0} N_2(\lambda) = |\xi(Ch_0)|^{1/(s-1)}.$$

The inequalities

$$\|Px(\lambda)\| \leqslant N_3|\xi[x(\lambda)]|^s \tag{6.78}$$

follow from (6.69). Therefore

$$\|x(\lambda) \leqslant \|Px(\lambda)\| + |\xi[x(\lambda)]| \leqslant$$
$$\leqslant N_3[N_2(\lambda)]^s|\lambda-\lambda_0|^{s/(s-1)} + N_2(\lambda)|\lambda-\lambda_0|^{1/(s-1)}$$

and

$$\|x(\lambda)\| \geqslant |\xi[x(\lambda)]| - \|Px(\lambda)\| \geqslant$$
$$\geqslant N_1(\lambda)|\lambda-\lambda_0|^{1/(s-1)} - N_3[N_2(\lambda)]^s|\lambda-\lambda_0|^{s/(s-1)},$$

that is

$$M_1(\lambda)|\lambda-\lambda_0|^{1/(s-1)} \leqslant \|x(\lambda)\| \leqslant M_2(\lambda)|\lambda-\lambda_0|^{1/(s-1)}, \tag{6.79}$$

where

$$\lim_{\lambda\to\lambda_0} M_1(\lambda) = \lim_{\lambda\to\lambda_0} M_2(\lambda) = |\xi CCh_0)|^{1/(s-1)}.$$

The inequalities (6.78) and (6.79) are the inequalities (6.55) and (6.56) respectively.

Thus, the proof of Theorem 6.12 will be complete if we show that the non-zero small solutions $x(\lambda)$ (for those values λ for which they exist) are unique for a given sign $\xi[x(\lambda)]$ and if the continuous dependence of $x(\lambda)$ on λ is proved.

6.4.6. *Lemma about the positiveness of an operator*

We assume that the spectrum of the operator $\boldsymbol{B}$, considered on E_0, lies in a circle whose radius is less that $k_0\lambda_0$ where $k_0<1$.

By virtue of Lemma 2.2, a norm $\|x\|_0$ equivalent to the one originally given can be introduced in E_0 such that

$$\|\boldsymbol{B}x\|_0 \leqslant k_0\lambda_0\|x\|_0 \qquad (x\in E_0). \tag{6.80}$$

In the entire space E, norms can be introduced which coincide with $\|x\|_0$ on E_0; it is possible to define them, for example, by the equality

$$\|x\|_1 = \|\boldsymbol{P}x\|_0 + |\xi(x)| \qquad (x\in E)$$

or

$$\|x\|_2 = \sqrt{\|\boldsymbol{P}x\|_0^2 + |\xi(x)|^2} \qquad (x\in E).$$

Additional arguments do not depend on how the norm $\|x\|_0$ is extended from E_0 to E; the notation $\|x\|_0$ is retained below for the extented norm. We shall assume that $\|h_0\|_0 = 1$.

If $\boldsymbol{B}$ is a self-adjoint operator in a Hilbert space, then a transition to a new norm is not necessary since the norm of a self-adjoint operator is equal to the radius of the smallest circle in which the spectrum lies.

We denote by $K(h_0, \varrho)$ (see page 35) the solid cone, allowing plastering, singled out by the conditions (1.20):

$$\xi(x)\geqslant 0, \quad \|\boldsymbol{P}x\|_0 \leqslant \varrho\xi(x), \tag{6.81}$$

where ϱ is some positive number. The inequality

$$\|x\|_0 \leqslant \|Px\|_0 + \|\xi(x)h_0\|_0 \leqslant (1+\varrho)\xi(x) \qquad (x \in K(h_0, \varrho)) \tag{6.82}$$

evidently follows from (6.81).

The operator B leaves every cone $K(h_0, \varrho)$ invariant. This property carries over to the operator A for the elements $x \in K(h_0, \varrho)$ with a small norm. The intersection of the cone $K(h_0, \varrho)$ with the sphere $\|x_0\| \leqslant r$ is denoted below by $T(\varrho; r)$.

Lemma 6.5. *With respect to all positive numbers ϱ_1 and ϱ_2 there can be constructed a number $r_0 > 0$ such that*

$$AT(\varrho; r_0) \subset K(h_0, \varrho)$$

for $\varrho_1 \leqslant \varrho \leqslant \varrho_2$. Moreover, a number r_0 can be chosen such that $Ax (x \in T(\varrho; r_0), x \neq \theta)$ are interior elements of the cone $K(h_0, \varrho)$.

Proof. We choose the number r_0 in such a way that for $\|x\|_0 \leqslant r_0$ the inequality

$$\|Cx + Dx\|_0 \leqslant \delta_0 \|x\|_0 \tag{6.83}$$

is satisfied, where δ_0 is so small that

$$\lambda_0 - \|\xi\| \delta_0 (1 + \varrho_2) > 0 ,$$

and for $\varrho_1 \leqslant \varrho \leqslant \varrho_2$,

$$\frac{k_0 \lambda_0 \varrho + \delta_0 \|P\|(1+\varrho)}{\lambda_0 - \|\xi\| \delta_0 (1+\varrho_2)} = \varrho^* < \varrho . \tag{6.84}$$

Let $x \in T(\varrho, r_0)$ and $x \neq \theta$. It is obvious that

$$\xi(Ax) = \xi(Bx) + \xi(Cx + Dx) \geqslant \lambda_0 \xi(x) - \|\xi\| \, \|Cx + Dx\|_0 ,$$

from which by virtue of (6.82) and (6.83)

$$\xi(Ax) \geqslant [\lambda_0 - \|\xi\| \delta_0 (1 + \varrho_2)] \xi(x) . \tag{6.85}$$

The inequality obtained means that for the elements

$$Ax (x \in T(\varrho, r_0), \; x \neq \theta, \; \varrho_1 \leqslant \varrho \leqslant \varrho_2)$$

the condition

$$\xi(Ax) > 0$$

is satisfied. Analogously,

$$\|PAx\|_0 \leqslant \|BPx\|_0 + \|P(Cx + Dx)\|_0 \leqslant k_0 \lambda_0 \|Px\|_0 + \delta_0 \|P\| \, \|x\|_0 ,$$

from which it follows that

$$\|PAx\|_0 \leqslant [k_0 \lambda_0 \varrho + \delta_0 \|P\| (1 + \varrho)] \xi(x)$$

and, by virtue of (6.85),

$$\|PAx\|_0 \leqslant \frac{k_0\lambda_0\varrho+\delta_0\|P\|(1+\delta)}{\lambda_0-\|\xi\|\delta_0(1+\varrho_2)}\xi(Ax)\,.$$

By virtue of (6.84), the last inequality signifies that the element Ax is a non-zero element of the cone $K(h_0, \varrho^*)$ and, consequently, an interior element of the cone $K(u_0, \varrho)$.
This completes the proof of the lemma.

We remark that the lemma is valid without the assumptions about the properties of the operators C and D. It suffices that B is a strong Fréchet derivative $A'(\theta)$ of the operator A.

6.4.7. *Lemma about the monotoneity of an operator*
Partial orderings, generated by distinct cones $K(h_0, \varrho)$, are utilized below
We write $x \overset{\varrho}{\leqslant} y$ if $y-x \in K(h_0, \varrho)$.
Let the conditions of the last subsection be satisfied.

Lemma 6.6. *With respect to the positive numbers ϱ_1 and $\varrho_2(\varrho_1<\varrho_2)$ a number $r_0>0$ can be found such that the operator A is monotone on the solid sphere $T\{\|x\| \leqslant r_0\}$ in the sense of a partial ordering generated by the cone $K(h_0, \varrho)(\varrho_1 \leqslant \varrho \leqslant \varrho_2)$.*

Proof. We choose a number r_0 so that for $\|x_1\|_0 \leqslant r_0$, $\|x_2\|_0 \leqslant r_0$, the inequality

$$\|Cx_1+Dx_1-Cx_2-Dx_2\|_0 \leqslant \delta_1\|x_1-x_2\|_0 \tag{6.86}$$

is satisfied where δ_1 is so small that

$$\lambda_0-\|\xi\|\delta_1(1+\varrho_2)>0\,, \tag{6.87}$$

and for $\varrho_1 \leqslant \varrho \leqslant \varrho_2$,

$$\frac{k_0\lambda_0\varrho+\|P\|\delta_1(1+\varrho_2)}{\lambda_0-\|\xi\|\delta_1(1+\varrho_2)}=\rho^{**}<\rho\,. \tag{6.88}$$

Let $x_1 \overset{\varrho}{\leqslant} x_2$.
From the equalities

$$Ax_1=Bx_1+Cx_1+Dx_1, \qquad Ax_2=Bx_2+Cx_2+Dx_2 \tag{6.89}$$

it follows that

$$\xi(Ax_2-Ax_1)=\lambda_0\xi(x_2-x_1)+\xi(Cx_2+Dx_2-Cx_1-Dx_1) \geqslant$$
$$\geqslant \lambda_0\xi(x_2-x_1)-\|\xi\|\delta_1\|x_2-x_1\|_0\,,$$

from which it follows that

$$\xi(Ax_2-Ax_1)\geqslant[\lambda_0-\|\xi\|\delta_1(1+\varrho_2)]\xi(x_2-x_1)\,. \tag{6.90}$$

It follows from the same equalities (6.89) that

$$\begin{aligned}\|P(Ax_2-Ax_1)\|_0&\leqslant\|BP(x_2-x_1)\|_0+\|P\|\,\|Cx_2+Dx_2-Cx_1-Dx_1\|_0\leqslant\\&\leqslant k_0\lambda_0\|P(x_2-x_1)\|_0+\|P\|\delta_1\|x_2-x_1\|_0\leqslant\\&\leqslant[k_0\lambda_0\varrho+\|P\|\delta_1(1+\varrho_2)]\xi(x_2-x_1)\,.\end{aligned}$$

from which by virtue of (6.90)

$$\|P(Ax_2-Ax_1)\|_0\leqslant\frac{k_0\lambda_0\varrho+\|P\|\delta_1(1+\varrho_2)}{\lambda_0-\|\xi\|\delta_1(1+\varrho_2)}\xi(Ax_2-Ax_1)\,.$$

By virtue of (6.88), the last inequality means that

$$Ax_2-Ax_1\in\dot{K}(h_0,\varrho)\,,\quad\text{that is}\quad Ax_1\overset{\varrho}{\leqslant}Ax_2\,.$$

This completes the proof of the lemma.

The lemma can be strenghtened, for example, in the following manner. Let $k_0\varrho_2<\varrho_1\leqslant\varrho_2$; then a number r_0 can be chosen such that the operator A is monotone on the sphere $\|x\|\leqslant r_0$ in a strengthened sense: it follows from $x_1\overset{\varrho_2}{\leqslant}x_2$ that $Ax_1\overset{\varrho_1}{\leqslant}Ax_2$. We also note that in the conditions of the lemma all the properties of the operators C and D are not utilized.

6.4.8. *Proof of uniqueness in the special case of a concave operator*
Lemma 6.7. *If*

$$\xi(Ch_0)<0\,, \tag{6.91}$$

then numbers ϱ_1 *and* $\varrho_2(0<\varrho_1\leqslant\varrho_2)$ *and a number* $r_0>0$ *can be found such that for every* $x\in T(\varrho_1, r_0)$ *and every* $t_0\in(0, 1)$ *a number* $\eta>0$ *can be found such that*

$$A(t_0x)\overset{\varrho_2}{\geqslant}(1+\eta)t_0Ax\,. \tag{6.92}$$

The proof is simple to complete but it requires cumbersome calculations. We choose a number ϱ_1 such that the inequality

$$\varrho_1(1+\varrho_1)^{s-1}\leqslant\frac{|\xi(Ch_0)|}{2q_0\|\xi\|} \tag{6.93}$$

will be satisfied where q_0 is the coefficient of condition (6.47). We choose a number ϱ_2 such that the inequality

$$\|PCh_0\|_0+q_0\|P\|\rho_1(1+\rho_1)^{s-1}<\frac{\varrho_2}{4}\,|\,\xi(Ch_0)| \tag{6.94}$$

is satisfied.

We stress that the number ϱ_2 can be assumed to be as large as desired. Let $x \in K(h_0, \varrho_1)$. Then

$$x = Px + \xi(x)h_0\,, \quad \xi(x) \geqslant 0, \quad \|P(x)\| \leqslant \varrho_1 \xi(x)\,.$$

It is obvious that

$$\|Cx - C[\xi(x)h_0]\|_0 \leqslant q_0(1+\varrho_1)^{s-1}|\xi(x)|^{s-1}\|Px\|_0 \leqslant q_0(1+\varrho_1)^{s-1}\varrho_1|\xi(x)|^s\,.$$

Therefore

$$\begin{aligned} -\xi(Cx) &= -[\xi(x)]^s\xi(Ch_0) + \xi\{C[\xi(x)h_0] - C\} \geqslant \\ &\geqslant [|\xi(Ch_0)| - q_0\|\xi\|(1+\varrho_1)^{s-1}\varrho_1][\xi(x)]^s \end{aligned}$$

and by virtue of (6.93)

$$-\xi(Cx) \geqslant \tfrac{1}{2}|\xi(Ch_0)|[\xi(x)]^s\,. \tag{6.95}$$

Analogously

$$\begin{aligned} \|PC_x\|_0 &\leqslant \|PC[\xi(x)h_0]\|_0 + \|P\{Cx - C[\xi(x)h_0]\}\|_0 \leqslant \\ &\leqslant [\|PCh_0\|_0 + \|P\|q_0(1+\varrho_1)^{s-1}\varrho_1][\xi(x)]^s \end{aligned}$$

and by virtue of (6.94)

$$\|PCx\|_0 \leqslant \frac{\varrho_2}{4}|\xi(Ch_0)|[\xi(x)]^s\,. \tag{6.96}$$

It follows from (6.95) and (6.96) that the elements $-Cx (x \neq \theta)$ belong to the cone $K(h_0, \varrho_2)$ for $x \in K(h_0; \varrho_1)$ together with the spherical neighborhood of radius $R_0\|x\|_0^s$ where

$$R_0 = \frac{\varrho_2|\xi(Ch_0)|}{4(\|P\| + \varrho_2\|\xi\|)(1+\varrho_1)^s}\,. \tag{6.97}$$

In fact, if

$$\|y\|_0 \leqslant R_0\|x\|_0^s\,,$$

then by virtue of (6.95)

$$\begin{aligned} \xi(-Cx+y) &\geqslant -\xi(Cx) - |\xi(y)| \geqslant \\ &\geqslant \frac{|\xi(Ch_0)|}{2}[\xi(x)]^s - \|\xi\| \cdot \|y\|_0 \geqslant \\ &\geqslant \frac{|\xi(Ch_0)|}{2} \cdot \frac{\|x\|_0^s}{(1+\varrho_1)^s} - \|\xi\| R_0\|x\|_0^s\,, \end{aligned}$$

from which it follows that

$$\xi(-Cx+y) \geqslant \frac{|\xi(Ch_0)|}{(1+\varrho_1)^s}\left[\tfrac{1}{2} - \frac{\varrho_2\|\xi\|}{4(\|P\| + \varrho_2\|\xi\|)}\right]\|x\|_0^s\,.$$

Analogously,

$$\|P(-Cx+y)\|_0 \leqslant \|PCx\|_0+\|Py\|_0 \leqslant$$
$$\leqslant \frac{\varrho_2}{4}\frac{|\xi(Ch_0)|}{(1+\varrho_1)^s}\|x\|_0^s+\|P\|R_0\|x\|_0^s \leqslant$$
$$\leqslant \frac{\varrho_2|\xi(Ch_0)|}{(1+\varrho_1)^s}\left[\tfrac{1}{4}+\frac{\|P\|}{4(\|P\|+\varrho_2\|\xi\|)}\right]\|x\|_0^s .$$

Therefore

$$\|P(-Cx+y)\|_0 \leqslant \varrho_2\xi(-Cx+y) .$$

We choose a number r_0 such that the inequalities

$$\|Dx\|_0 \leqslant \tfrac{1}{8}\left(1-\frac{1}{2^{s-1}}\right)R_0\|x\|_0^s \qquad (\|x\|_0 \leqslant r_0) \tag{6.98}$$

and

$$\|Dx_1-Dx_2\|_0 \leqslant \frac{R_0}{8}\|x_1-x_2\| \qquad (\|x_1\|_0,\ \|x_2\|_0 \leqslant r_0) \tag{6.99}$$

are satisfied.

Then for $\|x\| \leqslant r_0$ the inequalities

$$\|D(t_0x)-t_0Dx\|_0 \leqslant (t_0-t_0^s)\frac{R_0}{2}\|x\|_0^s \qquad (0 \leqslant t_0 \leqslant 1) \tag{6.100}$$

will be satisfied. In fact, for $0 \leqslant t_0 \leqslant \frac{1}{2}$, by virtue of (6.98),

$$\|D(t_0x)-t_0Dx\|_0 \leqslant \|D(t_0x)\|+t_0\|Dx\|_0 \leqslant$$
$$\leqslant \tfrac{1}{8}\left(1-\frac{1}{2^{s-1}}\right)R_0(t_0^s+t_0)\|x\|_0^s \geqslant$$
$$\leqslant \tfrac{1}{8}(1-t_0^{s-1})R_0\cdot 2t_0\|x\|_0^s ,$$

from which (6.100) follows. If $\frac{1}{2} \leqslant t_0 \leqslant 1$, then by virtue of (6.98) and (6.99)

$$\|D(t_0x)-t_0Dx\|_0 \leqslant (1-t_0)\|Dx\|_0+\|Dx-D(t_0x)\|_0 \leqslant$$
$$\leqslant (1-t_0)\tfrac{1}{8}\left(1-\frac{1}{2^{s-1}}\right)R_0\|x\|_0^s+\frac{R_0}{8}(1-t_0)^s\|x\|_0^s \leqslant$$
$$\leqslant \frac{1-t_0}{4}R_0\|x\|_0^s \leqslant \frac{t_0(1-t_0^s)}{2}R_0\|x\|_0^s ,$$

and again the inequality (6.100) is satisfied.

It follows from (6.100) that the elements

$$(-t_0+t_0^s)Cx-[t_0Dx-D(t_0x)] \qquad (x \in T(\varrho_1, r_0),\ 0 \leqslant t_0 \leqslant 1)$$

are contained in $K(h_0, \varrho_2)$ together with the spherical neighborhood of radius $(t_0-t_0^s)[R_0/2]\|x\|_0^s$.

We now fix $x \in T(\varrho_1, r_0)(x \neq \theta)$ and $t \in (0, 1)$. We choose a number $\eta > 0$ such that

$$t_0 \eta \| Bx + Cx + Dx \|_0 < (t_0 - t_0^s) \frac{R_0}{2} \| x \|_0^s ; \tag{6.101}$$

then

$$(-t + t_0^s)Cx - [t_0 Dx - D(t_0 x)] - \eta(t_0 Bx + t_0 Cx + t_0 Dx) \in K(h_0, \varrho_2) ,$$

that is,

$$C(t_0 x) + D(t_0 x) \overset{\varrho_2}{\geqslant} \eta t_0 Bx + (1 + \eta) t_0 (Cx + Dx) ,$$

from which it follows that

$$A(t_0 x) \overset{\varrho_2}{\geqslant} (1 + \eta) t_0 Ax .$$

This completes the proof of the lemma.

Lemmas 6.5, 6.6 and 6.7 mean that the operator A is monotone and h_0-concave on some $T(\varrho_1; r_0)$ in the sense of monotoneity generated by the cone $K(h_0, \varrho_2)$. It follows from Theorem 6.3 that the equation $Ax = \lambda x$ does not have more than one non-zero solution on $T(\varrho_1, r_0)$. By virtue of (6.78) there are no non-zero small solutions outside of $T(\varrho_1, r_0)$.
With this argument the proof of uniqueness will be complete for $\xi(Ch_0) < 0$ (for a special assumption about the spectrum of the operator B) of the non-zero solution $x(\lambda)$ for which $\xi[x(\lambda)] > 0$.

6.4.9. *Proof of uniqueness for the special case of a convex operator*
As in the three preceding subsections, we shall assume that the spectrum of the operator B, considered on E_0 lies in a circle whose radius is less than $k_0 \lambda_0$ where $k_0 < 1$.

Lemma 6.8. *Let condition* (6.91) *in Lemma* 6.7 *be replaced by the condition*

$$\xi(Ch_0) > 0 . \tag{6.102}$$

Then numbers ϱ_1 *and* $\varrho_2 (0 < \varrho_1 \leqslant \varrho_2)$ *and a number* $r_0 > 0$ *can be found such that for every* $x \in T(\varrho_1, r_0)$ *and every* $t_0 \in (0, 1)$ *a number* $\eta > 0$ *can be found such that*

$$A(t_0 x) \overset{\varrho_2}{\leqslant} (1 - \eta) t_0 Ax . \tag{6.103}$$

The proof is similar to the proof of Lemma 6.7. We choose the numbers ϱ_1 and ϱ_2 in such a way that the inequalities (6.93) and (6.94) are satisfied. Those arguments which were used in the proof of Lemma 6.7 bring us to

the conclusion that the elements Cx for $x \in K(h_0, \varrho_1)$ belong to the cone $K(h_0, \varrho_2)$ together with spherical neighborhood of radius $R_0\|x\|_0^s$, where R_0 is defined by the formula (6.97).
We further choose a number r_0 so that for $\|x\|_0 \leqslant r$ the inequality (6.100) will be satisfied. Then the elements

$$(t_0 - t_0^s)Cx + t_0 Dx - D(t_0 x)$$

are contained in $K(h_0, \varrho_2)$ together with a spherical neighborhood of radius $(t_0 - t_0^s)(R_0/2)\|x\|_0^s$.
For fixed $x \in T(\varrho_1; r_0)$ and $t_0 \in (0, 1)$ a number $\eta > 0$ can be found so that the inequality (6.100) is satisfied. Then

$$(t_0 - t_0^s)Cx + t_0 Dx - D(t_0 x) - \eta t_0(Bx + Cx + Dx) \in K(h_0, \varrho_2)\,,$$

that is,

$$C(t_0 x) + D(t_0 x) \overset{\varrho_2}{\leqslant} -\eta t_0 Bx + (1-\eta)t_0(Cx + Dx)\,,$$

from which (6.103) follows.
This completes the proof of the lemma.

Lemmas 6.5, 6.6 and 6.8 mean that the operator A is monotone and h_0-convex on $T(\varrho_1, r_0)$ under condition (6.102) in the sense of a partial ordering generated by the cone $K(h_0, \varrho_2)$. But then the uniqueness principle discussed in section 6.3 can be applied to equations with concave operators. In fact, if we assume that the equation $Ax = \lambda x$ has two non-zero solutions x_1 and x_2 on $T(\varrho_1; r_0)$ for some λ then it follows from (6.67) that either

$$\|P(x_1 - x_2)\| \leqslant \frac{q}{1-q}\,\xi(x_1 - x_2)$$

or

$$\|P(x_2 - x_1)\| \leqslant \frac{q}{1-q}\,\xi(x_2 - x_1)\,.$$

We can assume without loss of generality that $q/(1-q) < \varrho_2$. Then one of the elements $x_1 - x_2$ or $x_2 - x_1$ is either zero or an interior element of the cone $K(h_0; \varrho_2)$. It follows from the uniqueness principle that $x_1 = x_2$.
The cases considered in this and preceding subsections involve the following combination of the values $\xi(Ch_0)$ and λ:

$$\lambda > \lambda_0 \quad \text{and} \quad \xi(Ch) > 0$$

or

$$\lambda < \lambda_0 \quad \text{and} \quad \xi(Ch_0) < 0\,.$$

The remaining cases are easily reduced to these if the notations $h_1 = -h_0$, $\xi_1(x) = -\xi(x)$ are introduced.

6.4.10. *Proof of uniqueness in the general case*
Let a family $A(\beta)(|\beta-\beta_0| \leqslant \gamma_0)$ of non-linear operators be given each of which allows the representation

$$A(\beta) = B(\beta) + C(\beta) + D(\beta)\,, \tag{6.104}$$

where the $B(\beta)$ are linear operators each of which is defined by the equality

$$B(\beta)x = B(\beta)Dx + \lambda_0 \xi(x) h_0 \tag{6.105}$$

(the operator P of projection on E_0, the functional $\xi(x)$ and the element h_0 do not depend on β); the operators $C(\beta)$ are homogeneous of order s where the estimate

$$\|C(\beta)x_1 - C(\beta)x_2\| \leqslant q_0 r^{s-1} \|x_1 - x_2\| \qquad (\|x_1\|,\ \|x_2\| \leqslant r) \tag{6.106}$$

is valid in which q_0 does not depend on β. The operators $D(\beta)$ consist of terms of an order higher than s, where the estimates

$$\max_{\beta} \|D(\beta)x\| = o(\|x\|^s) \tag{6.107}$$

are valid and

$$\|D(\beta)x_1 - D(\beta)x_2\| \leqslant q_1(r)\|x_1 - x_2\| \qquad (\|x_1\|,\ \|x_2\| \leqslant r)\,, \tag{6.108}$$

where $q_1(r) = o(r^{s-1})$ and $q_1(r)$ does not depend on β.
Let $\lambda_0 > 0$ and the norms of all the operators $B(\beta)$ considered on E_0 be uniformly bounded by the number $k_0\lambda_0$, where $k_0 < 1$:

$$\|B(\beta)x\| \leqslant k_0 \lambda_0 \|x\| \qquad (x \in E_0\ ;\ |\beta - \beta_0| \leqslant \gamma_0)\,. \tag{6.109}$$

Finally, we assume that condition

$$\xi^* \leqslant |\xi[C(\beta)h_0]| \leqslant \xi^{**} \tag{6.110}$$

is satisfied where ξ^* and ξ^{**} are positive numbers.
Repeating the arguments developed in the proof of Lemmas 6.5–6.8, we convince ourselves of the existence of numbers ϱ_1 and ϱ_2 and $r_0 > 0$ such that on $T(\varrho_1, r_0)$ every operator $A(\beta)$ is monotone and either h_0-concave or h_0-convex (depending on the sign of $\xi[C(\varrho)h_0]$) in the sense of the partial ordering defined by the cone $K(h_0, \varrho_2)$. The number ϱ_2 can be assumed sufficiently large. All the numbers ϱ_1, ϱ_2, r_0 do not depend on β.
We turn to the study of equation $Ax = \lambda x$, where A allows the representation (6.45). The equation

$$Ax = \lambda x \tag{6.111}$$

is equivalent, obviously, to the system

$$\boldsymbol{BPx}+\boldsymbol{P}(\boldsymbol{Cx}+\boldsymbol{Dx})=\lambda\boldsymbol{Px}\,, \tag{6.112}$$

$$\lambda_0\xi(x)+\xi(\boldsymbol{Cx}+\boldsymbol{Dx})=\lambda\xi(x)\,. \tag{6.113}$$

The first of these equations is equivalent to the equation

$$\boldsymbol{R}(\lambda)\boldsymbol{P}(\boldsymbol{Cx}+\boldsymbol{Dx})=\boldsymbol{Px}\,, \tag{6.114}$$

where $\boldsymbol{R}(\lambda)$ considered on E_0 is the inverse operator to $\boldsymbol{B}-\lambda\boldsymbol{I}$. The system (6.113)–(6.114) is equivalent to one equation

$$\lambda\xi(x)h_0+\lambda\boldsymbol{R}(\lambda)\boldsymbol{P}(\boldsymbol{Cx}+\boldsymbol{Dx})+\xi(\boldsymbol{Cx}+\boldsymbol{Dx})h_0=\lambda x\,. \tag{6.115}$$

Thus, equation (6.111) is equivalent to the equation (6.115).
We consider the operators

$$\boldsymbol{A}(\lambda)=\lambda_0\xi(x)h_0+\lambda\boldsymbol{R}(\lambda)\boldsymbol{P}(\boldsymbol{Cx}+\boldsymbol{Dx})+\xi(\boldsymbol{Cx}+\boldsymbol{Dx})h_0\,. \tag{6.116}$$

These operators allow the representation (6.104) where

$$\boldsymbol{B}(\lambda)x=\lambda_0\xi(x)h_0\,, \tag{6.117}$$

$$\boldsymbol{C}(\lambda)x=\lambda\boldsymbol{R}(\lambda)\boldsymbol{PC}x+\xi(\boldsymbol{C}x)h_0\,, \tag{6.118}$$

$$\boldsymbol{D}(\lambda)x=\lambda\boldsymbol{R}(\lambda)\boldsymbol{PD}x+\xi(\boldsymbol{D}x)h_0\,. \tag{6.119}$$

All the properties enumerated in the beginning of this subsection follow from corresponding properties of the operators $\boldsymbol{C}$ and $\boldsymbol{D}$, since for values of λ near to λ_0, the norms of the operators $\boldsymbol{R}(\lambda)$ are uniformly bounded. Since $\lambda\boldsymbol{R}(\lambda)\boldsymbol{PC}h_0\in E_0$, then

$$\xi[\boldsymbol{C}(\lambda)h_0]=\xi[\xi(\boldsymbol{C}h_0)h_0]=\xi(\boldsymbol{C}h_0)\,.$$

This means that condition (6.110) is satisfied if $\xi(\boldsymbol{C}h_0)\neq 0$. Consequently, all the operators (6.116) are simultaneously λ_0-concave or h_0-convex on the same set $T(\varrho_1, r_0)$ for values of λ sufficiently near to λ_0 in the sense of a partial ordering with respect to a cone $K(h_0, \varrho_2)$ depending on what the sign of the number $\xi(\boldsymbol{C}h_0)$ is.
We assume that $\xi(\boldsymbol{C}h_0)<0$. Then the operators $\boldsymbol{A}(\lambda)$ are concave and, consequently, the equation $\boldsymbol{A}(\lambda)x=\lambda x$ does not have more than one non-zero solution on $T(\varrho_1, r_0)$. It follows from the equivalence of equations (6.115) and (6.111) that the equation (6.111) does not have more than one non-zero solution for values of λ near to λ_0 on $T(\varrho_1, r_0)$. It follows from (6.78) that all the solutions $x(\lambda)$ for which $\xi[x(\lambda)]>0$ lie in $T(\varrho_1, r_0)$. With this the proof of uniqueness is completed.

We assume that $\xi(Ch_0)>0$. The operators $A(\lambda)$ then are convex and from the uniqueness principle it follows that distinct solutions x^* and x^{**} of the equation $A(\lambda)=\lambda x$ (that is, equation $Ax=\lambda x$) have the property that x^*-x^{**} and $x^{**}-x^*$ are not interior elements of the cone $K(h_0, \varrho)$. This contradicts (6.67). Thus, for the case $\xi(Ch_0)>0$ the non-zero solution on $T(\varrho_1, r_0)$ is unique.

Cases when non-zero solutions $x(\lambda)$ exist for which $\xi[x(\lambda)]<0$ are reduced to the one examined if the replacement $h_0=h_1$, $\xi(x)=-\xi_1(x)$ is carried out.

6.4.11. *Continuous dependence on the parameter*

We assume that the equation $Ax=\lambda x$ has non-zero solutions $x(\lambda)$ ($\xi[x(\lambda)]>0$) for $\lambda\in(\lambda_0, \lambda_1)$ where λ_1 is sufficiently near to λ_0. We set $x(\lambda_0)=\theta$. By what has already been proved, the vector-function $x(\lambda)$ is single-valued.

The continuity of this function at the point $\lambda=\lambda_0$ follows from (6.79).

We consider the equation

$$y=-\boldsymbol{R}(\lambda)\boldsymbol{P}[\boldsymbol{C}(y+\alpha h_0)+\boldsymbol{D}(y+\alpha h_0)]\,, \tag{6.120}$$

where α is positive and sufficiently small, and λ is sufficiently near to λ_0. The values of the operator $\boldsymbol{S}(\alpha)$,

$$\boldsymbol{S}(\alpha)y=-\boldsymbol{R}(\lambda)\boldsymbol{P}[\boldsymbol{C}(y+\alpha h_0)+\boldsymbol{D}(y+\alpha h_0)]\,, \tag{6.121}$$

belong to E_0. Therefore, we look for the solutions of the equation (6.120) in E_0. It follows from the properties of the operators $\boldsymbol{C}$ and $\boldsymbol{D}$ that the operator $\boldsymbol{S}(\alpha)$ transforms some sphere $T_0\{\|y\|\leqslant r_0, y\in E_0\}$ into itself and on this sphere satisfies the contraction condition. The existence and uniqueness of solutions $y=y=(\lambda;\alpha)$ of equations (6.120) in the sphere T_0 follows from the principle of contraction mappings.

As is known, the resolvent $\boldsymbol{R}(\lambda)$ of the operator $\boldsymbol{B}$ in the subspace E_0 satisfies the operator equation

$$\boldsymbol{R}(\lambda)-\boldsymbol{R}(\mu)=(\lambda-\mu)\,\boldsymbol{R}(\lambda)\,\boldsymbol{R}(\mu)\,.$$

Therefore, for values λ_1 and λ_2 sufficiently near to λ_0 the inequality

$$\|\boldsymbol{R}(\lambda_1)-\boldsymbol{R}(\lambda_2)\|\leqslant K|\lambda_1-\lambda_2| \tag{6.122}$$

is satisfied. Since the operators $\boldsymbol{C}$ and $\boldsymbol{D}$ satisfy the Lipschitz condition with small coefficients on elements of a small norm, then it follows from (6.121) and (6.122) that

$$\|R(\lambda_1)P[C(y_1+\alpha_1 h_0)+D(y_1+\alpha_1 h_0)]-$$
$$-R(\lambda_2)P[C(y_2+\alpha_2 h_0)+D(y_2+\alpha_2 h_0)]\| \leqslant$$
$$\leqslant \|R(\lambda_1)-R(\lambda_2)\| \cdot \|P[C(y_1+\alpha_1 h_0)+D(y_1+\alpha_1 h_0)]\| +$$
$$+\|R(\lambda_2)\| \cdot \|P\| \{\|C(y_1+\alpha_1 h_0)-C(y_2+\alpha_2 h_0\| +$$
$$+\|D(y_1+\alpha_1 h_0)-D(y_2+\alpha_2 h_0)\|\} \leqslant$$
$$\leqslant K_1|\lambda_1-\lambda_2|+K_2\|y_1-y_2\|+K_3|\alpha_1-\alpha_2| , \tag{6.123}$$

where the numbers K_2 and K_3 can be assumed small. It follows from the inequalities obtained that the solution $y(\lambda, \alpha)$ of the equation (6.120) satisfies the Lipschitz condition with respect to the set of all variables. In fact, from

$$y(\lambda_1, \alpha_1) = -R(\lambda_1)P\{C[y(\lambda_1, \alpha_1)+\alpha_1 h_0]+D[y(\lambda_1, \alpha_1)+\alpha_1 h_0]\}$$

and

$$y(\lambda_2, \alpha_2) = -R(\lambda_2)P\{C[y(\lambda_2, \alpha_2)+\alpha_2 h_0]+D[y(\lambda_2, \alpha_2)+\alpha_2 h_0]\} ,$$

and from equality (6.123), it follows that

$$\|y(\lambda_1, \alpha_1)-y(\lambda_1, \alpha_2)\| \leqslant$$
$$\leqslant K_1|\lambda_1-\lambda_2|+K_2\|y(\lambda_1, \alpha_1)-y(\lambda_2, \alpha_2)\|+K_3|\alpha_1-\alpha_2| ,$$

from which it follows that

$$\|y(\lambda_1, \alpha_1)-y(\lambda_2, \alpha_2)\| \leqslant \frac{K_1}{1-K_2}|\lambda_1-\lambda_2| + \frac{K_3}{1-K_2}|\alpha_1-\alpha_2| . \tag{6.124}$$

If $x(\lambda)(\xi[x(\lambda)]>0)$ is a solution of the equation $Ax=\lambda x$, then

$$BPx(\lambda)+P\{C[Px(\lambda)+\xi[x(\lambda)]h_0]+D[Px(\lambda)+\xi[x(\lambda)]h_0]\}=\lambda Px(\lambda) .$$

Consequently,

$$Px(\lambda)=y\{\lambda, \xi[x(\lambda)]\} , \tag{6.125}$$

where $y(\lambda, \alpha)$ is a solution of equation (6.120). It follows from (6.124) that the vector-function $Px(\lambda)$ is continuous with respect to λ if the scalar function $\xi(\lambda)=\xi[x(\lambda)]$ is continuous. In this connection, the vector-function

$$x(\lambda)=Px(\lambda)+\xi(\lambda)h_0$$

is continuous.

We assume that $\xi(\lambda)$ is discontinuous at some point λ^*. Then a sequence $\lambda_n \to \lambda^*$ can be found such that the numbers $\xi_n=\xi(\lambda_n)$ converge to some number ξ^* which is different from $\xi(\lambda^*)$. It follows from (6.125) that the

elements $\boldsymbol{P}x(\lambda_m)$ will converge with respect to the norm to $y^*=y(\lambda^*, \xi^*)$. We pass in the equalities

$$\boldsymbol{B}\boldsymbol{P}x(\lambda_n)+\boldsymbol{P}\{\boldsymbol{C}[\boldsymbol{P}x(\lambda_n)+\xi_n h_0]+\boldsymbol{D}[\boldsymbol{P}x(\lambda_n)+\xi_n h_0]\}=\lambda_n \boldsymbol{P}x(\lambda_n)$$

and

$$\lambda_0\xi[x(\lambda_n)]+\xi\{\boldsymbol{C}[\boldsymbol{P}x(\lambda_n)+\xi_n h_0]+\boldsymbol{D}[\boldsymbol{P}x(\lambda_n)+\xi_n h_0]\}=\lambda_n\xi[x(\lambda_n)]$$

to the limit as $n\to\infty$. The limit equalities have the form

$$\boldsymbol{B}y^*+\boldsymbol{P}\{\boldsymbol{C}(y^*+\xi^* h_0)+\boldsymbol{D}(y^*+\xi^* h_0)\}=\lambda^* y^*$$

and

$$\lambda_0\xi^*+\xi\{\boldsymbol{C}(y^*+\xi^* h_0)+\boldsymbol{D}(y^*+\xi^* h_0)\}=\lambda^*\xi^* .$$

It follows from these equalities that

$$\boldsymbol{B}(y^*+\xi^* h_0)+\boldsymbol{C}(y^*+\xi^* h_0)+\boldsymbol{D}(y^*+\xi^* h_0)=\lambda^*(y^*+\xi^* h_0) ,$$

that is,

$$\boldsymbol{A}(y^*+\xi^* h_0)=\lambda^*(y^*+\xi^* h_0) .$$

Consequently

$$y^*+\xi^* h_0=x(\lambda^*) ,$$

from which the equality

$$\xi^*=\xi(\lambda^*)=\xi[x(\lambda^*)]$$

follows. We have arrived at a contradiction. This means that $\xi(\lambda)$ is continuous.

Thus, Theorem 6.12 is completely proved.

Chapter 7

APPLICATIONS

Applications of the general theory which was developed in preceding chapters are basic for the reduction of various problems to equations which have operators acting in Banach spaces with a cone and leave this cone invariant. In many cases, as it is seen from the sequel, such a reduction is not connected with the overcoming of essential difficulties.
In all the examples cited, we did not strive for the most general formulations. We were concerned only with basic results. More general statements can be obtained if in the consideration of the existing operators other criteria of monotoneity, positiveness, concavity, differentiability and so on are used.

7.1. Existence of positive solutions for integral equations

7.1.1. *Linear integral operators*

In this section, Ω denotes a bounded closed* set of a finite-dimensional space.
We consider a linear integral operator

$$Ax(t) = \int_{\Omega} k(t, s)x(s)ds \tag{7.1}$$

with a non-negative kernel $k(t, s)$. We shall assume that the kernel has properties by virtue of which the operator (7.1) acts and is continuous in some space E of functions which are defined on Ω. If E is the space C of continuous functions on Ω then a sufficient condition for complete continuity of the operator (7.1) in E is, for example, the continuity of the kernel $k(t, s)$. If E is the space $L_p(1 < p < \infty)$, then a sufficient condition for complete continuity is the inequality

* The assumption about the domain is not essential if the integral operators are considered in the spaces L_p.

$$\int_\Omega \int_\Omega |k(t, s)|^q \, dt \, ds < \infty , \tag{7.2}$$

where $(1/p)+(1/q)=1$. Less restricting but more clumsily formulated conditions for continuity and complete continuity of an integral operator in spaces L_p, in the space C, in Orlicz spaces and others (see the cited literature) are known.

It follows from the non-negativeness of the kernel $k(t, s)$ that the operator (7.1) leaves the cone K of non-negative functions in the space E invariant. In order that the theorems which were proved in chapter 2 can be considered as statements about equations with an integral operator (7.1), it is necessary to explain under what additional conditions this operator is u_0-positive or u_0-bounded above or below. Such additional conditions are not difficult to produce in the form of various inequalities which the kernel $k(t, s)$ or some iteration of it must satisfy. We restrict ourselves to the conditions which are sufficient for u_0-positiveness.

Let the kernel $k(t, s)$ satisfy the inequalities

$$u_0(t)\varphi(s) \leqslant k(t, s) \leqslant u_0(t)\psi(s) \qquad (t, s \in \Omega) , \tag{7.3}$$

where $u_0(t)$ is a non-negative function of the space E in which the operator (7.1) acts, and the functions $\varphi(s)$ and $\psi(s)$ become zero only on a set of measure zero and have the property that

$$\left| \int_\Omega x(s)\varphi(s) \, ds \right| < \infty , \quad \left| \int_\Omega x(s)\psi(s) \, ds \right| < \infty$$

for $x(t) \in E$. Then for every function $x(t) \in E$ the inequalities

$$\alpha u_0(t) \leqslant \int_\Omega k(t, s)x(s) \, ds \leqslant \beta u_0(t) \qquad (t \in \Omega) ,$$

where

$$\alpha = \int_\Omega \varphi(s)x(s) \, ds , \quad \beta = \int_\Omega \psi(s)x(s) \, ds$$

will be satisfied. Consequently, the inequalities (7.3) guarantee the u_0-positiveness of the operator (7.1).

We assume that the kernel $k(t, s)$, as this often happens, is "not bounded on the diagonal", that is,

$$\lim_{t \to s} k(t, s) = \infty \qquad (s \in \Omega) .$$

In this case the kernel $k(t, s)$ cannot satisfy inequalities of the type (7.3).

The u_0-positiveness has to be established for the linear integral operators with such kernels either with respect to estimates for iterated kernels or with integral estimates for the same kernel.

Lemma 7.1. *Let operator* (7.1) *act in a space C of continuous functions. Let for every set* $\Omega_1 \subset \Omega$ *of positive measure a positive number* $\eta = \eta(\Omega_1)$ *be found such that*

$$\eta \int_{\Omega} k(t, s)\,ds \leqslant \int_{\Omega_1} k(t, s)\,ds \qquad (t \in \Omega)\,. \tag{7.4}$$

Then the operator (7.1) *is* u_0*-positive where*

$$u_0(t) = \int_{\Omega} k(t, s)\,ds\,. \tag{7.5}$$

Moreover, for every non-negative function $x(t)$ *which is not identically equal to zero, the inequalities*

$$\alpha u_0(t) \leqslant \int_{\Omega} k(t, s)x(s)\,ds \leqslant \beta u_0(t) \qquad (t \in \Omega,\ \alpha, \beta > 0)$$

hold.

Proof. Let $x(t) \geqslant 0 (t \in \Omega)$ and $x(t) \not\equiv 0$. Then a set $\Omega_1 \subset \Omega$ can be found such that mes $\Omega_1 > 0$ and $x(t) \geqslant \alpha_0 > 0$ for $t \in \Omega_1$. Consequently,

$$Ax(t) = \int_{\Omega} k(t, s)x(s)\,ds \geqslant \alpha_0 \int_{\Omega_1} k(t, s)\,ds \geqslant \alpha_0 \eta(\Omega_1) u_0(t) (t \in \Omega)\,,$$

that is, A is u_0-bounded below.
The u_0-boundedness above is evident:

$$Ax(t) = \int_{\Omega} k(t, s)x(s)\,ds \leqslant \max_{s \in \Omega} |x(s)| u_0(t) \qquad (t \in \Omega)\,.$$

This completes the proof of the lemma.

It follows from the results of the second chapter, in particular, that for the satisfaction of the inequalities (7.3) or for the satisfaction of the conditions of Lemma 7.1, the integral operator (7.1) (completely continuous in some space E) has a unique non-negat ve normalized characteristic function to which there corresponds a positive value λ_0. The characteristic value λ_0 is simple. The remaining characteristic values are in absolute value strictly less than λ_0.

7.1.2. *Conditions for complete continuity of a non-linear integral operator* We consider the operator

$$Ax(t) = \int_{\Omega} k[t, s, x(s)]\,ds, \tag{7.6}$$

which in the sequel we shall call the *Uryson operator.*

In order that the operator (7.6) act and be completely continuous in the space C, the continuity of the function $k(t, s, u)(t, s \in \Omega, -\infty < u < \infty)$ with respect to the collection of variables is sufficient. The proof is obvious. (For more exact conditions for the complete continuity of the Uryson operator in the space C, see the cited literature.)

We assume that the Uryson operator does not act in the space C or it is inconvenient for some arguments to consider it in C. In this case, the spaces in which the Uryson operator can be considered are defined by the nature of the non-linearity of the function $k(t, s, u)$ with respect to the variable u. If these non-linearities are essentially non-power non-linearities, for example, exponential, then the Orlicz spaces are utilized. If the non-linearities are power non-linearities, then they will be by-passed by the spaces L_p.

We assume that

$$|k(t, s, u)| \leqslant R(t, s)(a + b|u|^{\alpha_0}), \tag{7.7}$$

where $\alpha_0 \geqslant 0$ and the function $R(t, s)$ is integrable with respect to both variables to some power $\beta_0 > 1$:

$$\int_{\Omega}\int_{\Omega} |R(t,s)|^{\beta_0}\,dt\,ds < \infty\,. \tag{7.8}$$

It is assumed that $k(t, s, u)$ is continuous with respect to u.

Theorem 7.1. *If*

$$\alpha_0 \leqslant \beta_0 - 1\,,$$

then the Uryson operator acts and is completely continuous in every space L_p *where* $p > 1$ *and*

$$\frac{\alpha_0\beta_0}{\beta_0 - 1} \leqslant p \leqslant \beta_0\,. \tag{7.9}$$

We do not go into the proof of this theorem.

We cited Theorem 7.1 for the case when the function $k(t, s, u)$ satisfies

condition (7.7) for all values of u. If these conditions are satisfied only for $u \geqslant 0$, then the Uryson operator is defined and is completely continuous on the cone K of non-negative functions in L_p. In order to convince ourselves of this, it suffices to consider the function

$$k_1(t, s, u) = \begin{cases} k(t, s, u), & \text{if } u \geqslant 0 \\ k(t, s, -u), & \text{if } u \leqslant 0 \end{cases}$$

and the operator

$$A_1 x(t) = \int_\Omega k_1[t, s, x(s)]\, ds\ ;$$

the function $k_1(t, s, u)$ will satisfy condition (7.7) for all u; the operator A_1 will, by virtue of Theorem 7.1, be completely continuous on all L_p; it remains to remark that the operators A and A_1 assume identical values on K.

Analogously, the Uryson operator is completely continuous on the cone K of non-negative functions of the space C if $k(t, s, u)$ is continuous with respect to all the variables $t, s \in \Omega$ and $u \geqslant 0$.

We have not given the conditions for complete continuity of the Uryson operator in an Orlicz space (see M. A. Krasnosel'skii and Ya. B. Rutickii [1]). We note, however, that for the condition

$$|k(t, s, u)| \leqslant \Phi(u) \qquad (t, s \in \Omega,\ -\infty < u < \infty)$$

the Ursyson operator is completely continuous in some Orlicz space L_M* where $M(u)$ is defined with respect to the function $\Phi(u)$.

7.1.3. *Differentiability of a Uryson operator*

We first mention simple conditions for differentiability according to Fréchet of a Uryson operator in the space C.

We assume that the function $k(t, s, u)$ is continuous together with its derivative $k_u'(t, s, u)$ with respect to all the variables $t, s \in \Omega, -\infty < u < \infty$. Then the Uryson operator is strongly differentiable according to Fréchet at every point $x_0(t)$ of the space C. The Fréchet derivative is defined by the formula

$$A'(x_0)h(t) = \int_\Omega k_u'[t, s, x_0(s)]\, h(s)\, ds\ . \tag{7.10}$$

The proof is obvious since, by virtue of the formula of finite increments,

$$A(x_0+h)-Ax_0-A'(x_0)h=$$
$$=\int_\Omega \{k_u'[t, s, x_0(s)+\theta(t, s)h(s)]-k_u'[t, s, x_0(s)]\}h(s)\,ds, \qquad (7.11)$$

where $0<\theta(t, s)<1$, from which it follows that

$$\frac{\|A(x_0+h)-Ax_0-A'(x_0)h\|}{\|h\|} \leqslant$$
$$\leqslant \text{mes}\,\Omega \sup_{t,s\in\Omega} |k_u'[t, s, x_0(s)+\theta(t, s)h(s)]-k_u'[t, s, x_0(s)]|\,,$$

and since $k_u'(t, s, u)$ is uniformly continuous with respect to u, then

$$\lim_{\|h\|\to 0} \frac{\|A(x_0+h)-Ax_0-A'(x_0)h\|}{\|h\|} = 0\,.$$

Unlike the case of space C, the differentiability of the Uryson operator in the space L_p does not follow from the continuous differentiability with respect to the variable u of the function $k(t, s, u)$. For example, the operator

$$Ax(t) = \int_\Omega \sin e^{x(s)}\,ds$$

acts and is completely continuous in an arbitrary space L_p. However, the operator (7.10) cannot be its derivative and, moreover, cannot be defined on L_p since in the example being considered

$$k_u'[t, s, x_0(s)]=e^{x_0(s)}\cos e^{x_0(s)}\,,$$

and this function might be non-integrable.

We give one example of the conditions for differentiability according to Fréchet.

Let the Uryson operator act in L_2. Then a sufficient condition for strong differentiability according to Fréchet at an arbitrary point $x_0=x_0(t)$ of the space L_2 is the continuity with respect to u of the derivative $k_u'(t, s, u)$ and the inequalities

$$|k_u'(t, s, u)|\leqslant a+b|u| \qquad (t, s\in\Omega,\ -\infty<u<\infty) \qquad (7.12)$$

are satisfied. In this connection, the derivative $A'(x)$ is defined by equality (7.10).

For the proof, we remark that by virtue of the Buniakovskii-Cauchy inequality the inequality

$$|A(x_0+h)-Ax_0-A'(x_0)h|\leqslant$$
$$\leqslant \|h\| \sqrt{\int_\Omega |k_u'[t, s, x_0(s)+\theta(t, s)h(s)]-k_u'[t, s, x_0(s)]|^2\,ds}$$

follows from (7.11) from which it follows that

$$\frac{\|A(x_0+h)-Ax_0-A'(x_0)h\|}{\|h\|} \leqslant$$

$$\leqslant \sqrt{\int_\Omega \int_\Omega |k_u'[t, s, x_0(s)+\theta(t, s)h(s)]-k_u'[t, s, x_0(s)]|^2 dt\, ds}\,.$$

It remains to show that the double integral in the right member approaches zero when $\|h\|\to 0$. But this is obvious since by virtue of the continuity of $k_u'(t, s, u)$ with respect to the variable the integrand expression approaches zero in measure and by virtue of (7.12) the functions

$$|k_u'[t, s, x_0(s)+\theta_n(t, s)h_n(s)]-k_u'[t, s, x_0(s)]|^2 \qquad (n=1, 2, \ldots)$$

have absolutely equi-continuous integrals for an arbitrary sequence $h_n(t)\in L_2$, $\|h_n\|\to 0$. Therefore, (see I. P. Natanson [1], page 168, Vitali's theorem) one can pass to the limit under the integral sign.

7.1.4. *Derivatives with respect to a cone*
We introduce for consideration the function

$$H(t, s, u) = \frac{1}{u} k(t, s, u) \qquad (u>0)\,. \tag{7.13}$$

We shall say that $H(t, s, u)$ *decreases almost everywhere* as u increases if this function does not increase with respect to u and if for an arbitrary fixed number $t\in\Omega$ and for arbitrary fixed $u_2>u_1$ the inequality

$$H(t, s, u_1)-H(t, s, u_2)>0 \tag{7.14}$$

is satisfied for almost all $s\in\Omega$.

Theorem 7.2. *Let* $k(t, s, 0)\equiv 0$ *and the derivative*

$$k_u'(t, s, 0)=P(t, s) \qquad (t, s\in\Omega) \tag{7.15}$$

exist. Let the operator (7.1) *and the linear integral operator*

$$Px(t) = \int_\Omega P(t, s)x(s)ds \tag{7.16}$$

act in the space C. Finally, let the function $H(t, s, u)$ *decrease almost everywhere as u increases (for small u).*
Then the operator (7.16) *is a strong Fréchet derivative* $A'(\theta)$, *with respect to the cone K of non-negative functions of the space C, of the operator* (7.6).

Proof. Let $x_n(t)$ $(n=1, 2, \ldots)$ be a sequence of continuous non-negative functions which converges uniformly to zero. Without loss of generality, it can be assumed that the $\|x_n\|$ $(n=1, 2, \ldots)$ monotonely decrease. We denote by Ω_n $(n=1, 2, \ldots)$ the set of those numbers $t\in\Omega$, in which $x_n>0$. It is obvious that

$$\frac{\|Ax_n-Px_n\|}{\|x_n\|} = \max_t \frac{1}{\|x_n\|}\int_\Omega \{P(t, s)x_n(s)-k[t, s, x_n(s)]\}ds= $$

$$= \max_t \frac{1}{\|x_n\|}\int_{\Omega_n} \{P(t, s)-H[t, s, x_n(s)]\}x_n(s)ds\leqslant$$

$$\leqslant \max_t \int_{\Omega_n} \{P(t, s)-H[t, s, x_n(s)]\}ds\,.$$

By virtue of the fact that the function $H(t, s, u)$ decreases almost everywhere, it follows from the last inequality that

$$\frac{\|Ax_n-Px_n\|}{\|x_n\|} \leqslant \max_t \int_\Omega \{P(t, s)-H[t, s, \|x_n\|]\}ds\,. \tag{7.17}$$

The functions

$$f_n(t) = \int_\Omega \{P(t, s)-H[t, s, \|x_n\|]\}ds \qquad (n=1, 2, \ldots)$$

are continuous since the operators (7.16) and (7.1) act in C. The integrand expressions for all $t, s\in\Omega$ approach zero. For every fixed number t we can pass to the limit under the integral sign since the non-negative integrand expressions monotonely decrease. Consequently, the $f_n(t)$ converge to zero and are monotonely decreasing. By virtue of the Dini theorem, the sequence $f_n(t)$ converges uniformly to zero.
Therefore, it follows from (7.17) that

$$\lim_{\|x_n\|\to 0} \frac{\|Ax_n-A0-Px_n\|}{\|x_n\|} = \lim_{\|x_n\|\to 0} \frac{\|Ax_n-Px_n\|}{\|x_n\|} = 0\,.$$

This completes the proof of the theorem.

The assumption about the decrease of the function $H(t, s, u)$ means that for fixed numbers $t, s\in\Omega$ the graph of the continuous function $H(t, s, u)$ for $u>0$ has the property such that every straight line $y=ku$ intersects it at not more than one point. If $k(t, s, u)$ is differentiable with respect to u, then $H(t, s, u)$ also is differentiable with respect to u for $u>0$, where

$$H_u'(t, s, u) = \frac{uk_u'(t, s, u)-k(t, s, u)}{u^2}\,.$$

Therefore, it suffices for the decrease of the function $H(t, s, u)$ that the inequality

$$uk_u'(t, s, u) < k(t, s, u) \tag{7.18}$$

is satisfied.

The inequality (7.18) evidently is satisfied if $k(t, s, 0) \geqslant 0$ and the derivative $k_u'(t, s, u)$ decreases since

$$k(t, s, u) = k(t, s, 0) + uk_u'[t, s, \theta(t, s)u] .$$

We cite an analogy of Theorem 7.2 for operators acting in the spaces L_p. The following inequality, which evidently follows from Hölder's inequality, will be used below:

$$\left\{\int_\Omega \left|\int_{\Omega_1} u(t, s)v(s)\,ds\right|^p dt\right\}^{1/p} \leqslant$$

$$\leqslant \|v(s)\|_{L_p} (\text{mes } \Omega)^{(|p-q|)/pq} \left\{\int_\Omega \int_{\Omega_1} |u(t, s)|^r\,dt\,ds\right\}^{1/r}, \tag{7.19}$$

where $\Omega_1 \subset \Omega$, $p > 1$, $q = \dfrac{p}{p-q}$, $r = \max\{p, q\}$.

Theorem 7.3. *Let the function $H(t, s, u)$ be non-negative and decrease as u increases. Let $k(t, s, 0) \equiv 0$. Finally, let the derivative* (7.15) *exist where*

$$\int_\Omega \int_\Omega |P(t, s)|^r\,dt\,ds < \infty . \tag{7.20}$$

Then the Uryson operator (7.6) *is defined on the cone K of non-negative functions of the space L_p; its values belong to L_p; the formula* (7.16) *defines the strong Fréchet derivative of the operator* (7.6) *at the point θ with respect to the cone K.*

Proof. We first show that it follows from (7.20) and from the decrease of the function $H(t, s, u)$ that the Uryson operator is defined on a non-negative function of L_p and the set of its values lies in L_p. In fact

$$P(t, s) = \lim_{u \to +0} \frac{k(t, s, u) - k(t, s, 0)}{u} = \lim_{u \to +0} H(t, s, u)$$

and by virtue of the decrease of $H(t, s, u)$

$$P(t, s) \geqslant H(t, s, u) \qquad (t, s \in \Omega, u \geqslant 0) .$$

Therefore, for an arbitrary function $x(t) \in L_p$ $(x(t) \geqslant 0)$ it follows from the

inequality (7.19) that

$$\left\{\int_\Omega \left|\int_\Omega k[t, s, x(s)]ds\right|^p dt\right\}^{1/p} \leqslant \left\{\int_\Omega \left|\int_\Omega P(t, s)x(s)ds\right|^p dt\right\}^{1/p} \leqslant$$
$$\leqslant \|x(s)\|_{L_p} (\text{mes } \Omega)^{(|p-q|)/pq} \left\{\int_\Omega \int_\Omega |P(t, s)|^r dt\, ds\right\}^{1/r} < \infty ,$$

that is, $Ax(t) \in L_p$.

Let $\varepsilon > 0$ be given.

By virtue of the absolute continuity of the integral (7.20), a number $\delta > 0$ can be found such that for an arbitrary set $\tilde{\Omega} \subset \Omega \times \Omega$ the inequality

$$\left\{\int_{\tilde{\Omega}} \int |P(t, s)|^r dt\, ds\right\}^{1/r} < \varepsilon \tag{7.21}$$

will follow from mes $\tilde{\Omega} < \delta$.

On the basis of Egorov's theorem, a set $\mathfrak{M}_\delta \subset \Omega \times \Omega$ can be found such that

$$\text{mes } [(\Omega \times \Omega) \setminus \mathfrak{M}_\delta] < \delta$$

and on $\mathfrak{M}_\delta$ the sequence of functions

$$nk\left(t, s, \frac{1}{n}\right) = H\left(t, s, \frac{1}{n}\right) \qquad (n = 1, 2, \ldots)$$

converges to $P(t, s)$ uniformly. We choose a natural number n_0 so that for $n \geqslant n_0$ the inequality

$$\left|nk\left(t, s, \frac{1}{n}\right) - P(t, s)\right| \leqslant \varepsilon , \qquad (\{t, s\} \in \mathfrak{M}_\delta) \tag{7.22}$$

is satisfied. Finally, we choose a natural number k such that

$$n_0^{-(k-1)p} < \frac{\delta}{\text{mes } \Omega} . \tag{7.23}$$

Let $x(t)$ be a non-negative function of L_p such that

$$\|x(t)\| \leqslant n_0^{-k} .$$

We denote by Ω_1 the set of those numbers $t \in \Omega$ in which the inequality

$$x(t) \geqslant \frac{1}{n_0}$$

is satisfied. Then

$$\frac{1}{n_0^k} \geqslant \|x(t)\| = \left\{\int_\Omega |x(t)|^p dt\right\}^{1/p} \geqslant \left\{\int_{\Omega_1} |x(t)|^p dt\right\}^{1/p} \geqslant \frac{1}{n_0} (\text{mes } \Omega_1)^{1/p} ,$$

from which, by virtue of (7.23), it follows that

$$\text{mes } \Omega_1 \leqslant \frac{1}{n_0^{(k-1)p}} < \frac{\delta}{\text{mes } \Omega}. \tag{7.24}$$

We denote by Ω_0 the set of those numbers $t \in \Omega$ in which $x(t) > 0$. Obviously

$$\frac{\|Px - Ax\|}{\|x\|} =$$

$$= \frac{1}{\|x\|} \left\{ \int_\Omega \left| \int_{\Omega_1} \{P(t, s) - H[t, s, x(s)]\} x(s)\, ds \right|^p dt \right\}^{1/p} \leqslant I_1 + I_2,$$

where

$$I_1 = \frac{1}{\|x\|} \left\{ \int_\Omega \left| \int_{\Omega_0} \{P(t, s) - H[t, s, x(s)]\} x(s)\, ds \right|^p dt \right\}^{1/p}$$

and

$$I_2 = \frac{1}{\|x\|} \left\{ \int_\Omega \left| \int_{\Omega_0 / \Omega_1} \{P(t, s) - H[t, s, x(s)]\} x(s)\, ds \right|^p dt \right\}^{1/p}.$$

We evaluate separately each term I_1 and I_2.

The inequality (7.19), applied to the first term, yields

$$I_1 \leqslant (\text{mes } \Omega)^{(|p-q|)/pq} \left\{ \int_\Omega \int_{\Omega_1} |P(t, s) - H[t, s, x(s)]|^r\, dt\, ds \right\}^{1/r},$$

from which it follows that

$$I_1 \leqslant (\text{mes } \Omega)^{(|p-q|)/pq} \left\{ \int_\Omega \int_{\Omega_1} |P(t,s)|^r\, dt\, ds \right\}^{1/r}$$

and, by virtue of (7.24) and (7.21),

$$I_1 \leqslant \varepsilon (\text{mes } \Omega)^{(|p-q|)/pq}.$$

We introduce the notation

$$\mathfrak{M}_1 = \mathfrak{M}_\varrho \cap [\Omega \times (\Omega_0 \setminus \Omega_1)], \quad \mathfrak{M}_2 = [\Omega \times (\Omega_0 \setminus \Omega_1)] \setminus \mathfrak{M}_1.$$

The inequality (7.19), applied to I_2 yields

$$I_2 \leqslant (\text{mes } \Omega)^{(|p-q|)/pq} \left\{ \int_{\mathfrak{M}_1} \int |P(t, s) - H[t, s, x(s)]|^r\, dt\, ds + \right.$$

$$\left. + \int_{\mathfrak{M}_2} \int |P(t, s) - H[t, s, x(s)|^r\, dt\, ds \right\}^{1/r}.$$

It is easy to see that mes $\mathfrak{M}_2 < \delta$. Therefore, by virtue of (7.21),

$$\int_{\mathfrak{M}_2} \int |P(t, s) - H[t, s, x(s)]|^r\, dt\, ds \leqslant \int_{\mathfrak{M}} \int |P(t, s)|^r\, dt\, ds \leqslant \varepsilon^r.$$

By virtue of (7.22),

$$\int_{\Omega_1}\int |P(t,s)-H[t,s,x(s)]|^r\,dt\,ds \leqslant$$

$$\leqslant \int_{\Omega_1}\int \left|P(t,s)-n_0 k\left(t,s,\frac{1}{n_0}\right)\right|^r dt\,ds \leqslant \varepsilon^r(\text{mes }\Omega)^2 .$$

It follows from the last two estimates that

$$I_2 \leqslant \varepsilon(\text{mes }\Omega)^{(|p-q|)/pq}[1+(\text{mes }\Omega)^2]^{1/r} .$$

Thus, for $\|x(t)\| \leqslant n_0^{-k}$ the inequality

$$\frac{\|Px-Ax\|}{\|x\|} \leqslant \varepsilon\ (\text{mes }\Omega)^{(|p-q|)/pq}\{1+[1+(\text{mes }\Omega)^2]^{1/r}\}$$

is satisfied. It follows from here that

$$\lim_{\|x\|\to 0,\ x\in K} \frac{\|Px-Ax\|}{\|x\|} = 0 .$$

This completes the proof of the theorem.

7.1.5. *Asymptotic derivatives*

Let the Uryson operator be defined on the cone K of non-negative functions in the space C or the space $L_p(1<p<\infty)$ and be completely continuous in a corresponding space. We assume that the function

$$H(t,s,u) = \frac{1}{u}k(t,s,u)$$

as $u\to+\infty$ approaches uniformly with respect to t, $s\in\Omega$ to some function $Q(t,s)$ $(t,s\in\Omega)$, where the integral operator

$$Qx(t) = \int_\Omega Q(t,s)x(s)\,ds \tag{7.25}$$

acts in a corresponding space. It turns out that Q is a *strong asymptotic derivative with respect to the cone K of the Uryson operator* (7.6).

We carry out the proof for the case of the space C. Let $\varepsilon>0$ be given. We choose a number R such that for $u\geqslant R$ the inequality

$$|H(t,s,u)-Q(t,s)|<\varepsilon \qquad (t,s\in\Omega) \tag{7.26}$$

is satisfied. We denote by K_R the intersection of the cone K with the sphere $\|x\|\leqslant R$. Let

$$M = \sup_{x\in K_R}\{\|Ax(t)\|,\ \|Qx(t)\|\} ; \tag{7.27}$$

the number M is finite since the sets AK_R and QK_R are compact.
We introduce for consideration the operators U_1 and U_2 by the equalities

$$U_1x(t) = \begin{cases} x(t), & \text{if} \quad x(t) \leqslant R, \\ R, & \text{if} \quad x(t) \geqslant R, \end{cases}$$

and

$$U_2x(t) = \begin{cases} R, & \text{if} \quad x(t) \leqslant R, \\ x(t), & \text{if} \quad x(t) \geqslant R. \end{cases}$$

Obviously, for every function $x(t) \in K$,

$$Ax(t) = \int_\Omega k[t, s, x(s)]\,ds =$$

$$= \int_\Omega k[t, s, U_1x(s)]ds + \int_\Omega k[t, s, U_2x(s)]ds - \int_\Omega k[t, s, R]\,ds =$$

$$= AU_1x(t) + AU_2x(t) - AR(t),$$

where $R(t) \equiv R$ and

$$Qx(t) = \int_\Omega Q(t, s)U_1x(s)ds + \int_\Omega Q(t, s)U_2x(s)ds - R\int_\Omega Q(t, s)\,ds =$$

$$= QU_1x(t) + QU_2x(t) - QR(t).$$

By virtue of (7.27),

$$\|AU_1x(t)\| \leqslant M, \quad \|AR(t)\| \leqslant M, \quad \|QU_1x(t)\| \leqslant M, \quad \|QR(t)\| \leqslant M.$$

Therefore,

$$\|Ax(t) - Qx(t)\| \leqslant 4M + \|AU_2x(t) - QU_2x(t)\| =$$

$$= 4M + \left\| \int_\Omega \{H[t, s, U_2x(s)] - Q(t, s)\}U_2x(s)\,ds \right\|$$

and by virtue of (7.26)

$$\|Ax(t) - Qx(t)\| \leqslant 4M + \varepsilon \left\| \int_\Omega U_2x(s)\,ds \right\| =$$

$$= 4M + \varepsilon(\text{mes } \Omega)^{1/p} \int_\Omega U_2x(s)\,ds,$$

from which it follows that

$$\|Ax(t) - Qx(t)\| \leqslant 4M + \varepsilon(\text{mes } \Omega)^{1/p} \int_\Omega [R + x(s)]\,ds \leqslant$$

$$\leqslant 4M + \varepsilon(a + b\|x\|).$$

It follows from the last inequality that

$$\overline{\lim_{\|x\|\to\infty,\ x\in K}} \frac{\|Ax(t)-Qx(t)\|}{\|x\|} \leqslant b\varepsilon$$

and, by virtue of the arbitrariness of ε, that

$$\lim_{\|x\|\to\infty,\ x\in K} \frac{\|Ax(t)-Qx(t)\|}{\|x\|} = 0\,.$$

With this the proof is completed.

For the case of the space L_p the argument almost does not change.

7.1.6. *Positiveness of an integral operator*

From now on, it will be assumed that the condition

$$k(t, s,u)\geqslant 0 \qquad (t, s\in\Omega,\ u\geqslant 0) \tag{7.28}$$

is satisfied. Then the Uryson operator (7.6) will leave the cone K of non-negative functions in the space E in which it acts invariant.

For some cases, more restricted cones, which are invariant for a Uryson operator, can be found.

Let the additional condition

$$\sup_{t\in\Omega} k(t, s, u)\leqslant \varrho \inf_{t\in\Omega} k(t, s, u) \qquad (s\in\Omega,\ u\geqslant 0) \tag{7.29}$$

be satisfied besides (7.28) where the fixed number $\varrho>1$. If, for example,

$$k(t, s, u)=k(t, s)f(s, u) \qquad (t, s\in\Omega,\ u\geqslant 0) \tag{7.30}$$

and the kernel $k(t, s)$ is continuous and positive, then condition (7.29) is satisfied. In fact, if

$$M=\max_{t,\ s\in\Omega} k(t, s), \qquad m=\min_{t,\ s\in\Omega} k(t, s)\,,$$

then

$$\sup_{t\in\Omega} k(t, s, u)\leqslant Mf(s, u) \leqslant \frac{M}{m}\inf_{t\in\Omega} k(t, s, u)\,.$$

Let the Uryson operator be defined on the cone K of non-negative functions of the space C or one of the spaces $L_p(1<p<\infty)$. In the same space, we consider the set $\tilde{K}$ of the non-negative functions $x(t)$ satisfying the additional condition

$$\sup_{t\in\Omega} x(t)\leqslant \varrho \inf_{t\in\Omega} x(t)\,.$$

The set $\tilde{K}$ is the cone $K_{u_0,\varrho}$ (see page 47) where $u_0(t)\equiv 1$. Since the cone K

of non-negative functions in the space C and in the space L_p is normal, then the cone $\tilde{K}$ by virtue of Theorem 1.13 allows plastering.

If condition (7.29) *is satisfied, then the Uryson operator leaves the cone* $\tilde{K}$ *invariant*; moreover, the Uryson operator under this condition transforms all the non-negative functions into elements of $\tilde{K}$. For the proof, we consider the arbitrary non-negative function $x(t)$; by virtue of (7.29), for arbitrary $t_1, t_2 \in \Omega$, the inequalities

$$Ax(t_1) = \int_\Omega k[t_1, s, x(s)]\,ds \leqslant \varrho \int_\Omega k[t_2, s, x(s)]\,ds = \varrho Ax(t_2)\,,$$

hold from which if follows that

$$\sup_{t_1 \in \Omega} Ax(t_1) \leqslant \varrho Ax(t_2)$$

and

$$\sup_{t_1 \in \Omega} Ax(t_1) \leqslant \varrho \inf_{t_2 \in \Omega} Ax(t_2)\,.$$

7.1.7. *Non-negative solutions*

Let the conditions (see subsection 7.1.2) be satisfied for which the Uryson operator is defined and completely continuous on the cone K of non-negative functions of one of the spaces C or L_p. Moreover, let the operator A have a strong asymptotic derivative with respect to a cone where

$$A'(\infty)x(t) = \int_\Omega Q(t, s)x(s)\,ds\,. \tag{7.31}$$

We assume that condition (7.28) is satisfied; then the kernel $Q(t, s)$ will be non-negative and the operator $A'(\infty)$ will be a positive linear completely continuous operator acting in a corresponding functional space.

Theorem 4.7 implies the next theorem.

Theorem 7.4. *Under the enumerated conditions, for the existence for a non-linear integral equation*

$$x(t) = \int_\Omega k[t, s, x(s)]\,ds \tag{7.32}$$

of at least one non-negative solution, it suffices that the kernel $Q(t, s)$ *have positive characteristic values which are greater than or equal to* 1.

A simpler condition for the existence of a non-negative solution for equation (7.32) can appear in the form of the inequalities

$$0 \leqslant k(t, s, u) \leqslant a + bu \qquad (t,\ s \in \Omega,\ u \geqslant 0)\,, \tag{7.33}$$

where $a \geqslant 0$ and

$$b \text{ mes } \Omega < 1\,. \tag{7.34}$$

If A acts in the space C of continuous functions, then the proof of the formulated statement is obvious. It suffices to consider the intersection K_R of the cone of non-negative functions with the sphere $\|x\| \leqslant R$ where

$$R \geqslant \frac{a \text{ mes } \Omega}{1 - b \text{ mes } \Omega}\,,$$

and remark that a completely continuous operator A transforms K_R into itself since for $x(t) \in K_R$

$$Ax(t) \leqslant \int_\Omega [a + bx(t)]\,dt = a \text{ mes } \Omega + b \int_\Omega x(t)\,dt \leqslant$$

$$\leqslant R(1 - b \text{ mes } \Omega) + bR \text{ mes } \Omega = R\,.$$

After this it remains to use the Schauder principle. For the case when A acts in L_p the proof is analogous: A transforms K_R into itself where

$$R \geqslant \frac{a(\text{mes } \Omega)^{1+(1/p)}}{1 - b \text{ mes } \Omega}\,.$$

The existence of a non-negative solution for condition (7.13) can be ob tained as a corollary to Theorem 4.9.

7.1.8. *Positive solutions*

In this subsection, it is assumed that

$$k(t, s, 0) \equiv 0 \qquad (t, s \in \Omega)\,. \tag{7.35}$$

In this connection, the equation (7.30) has the trivial solution. We mention some conditions following from Theorems 4.11–4.16 for the existence of a second (already non-trivial) non-negative solution.

We shall assume that condition (7.28) is satisfied, from which it follows that the Uryson operator transforms the cone K of non-negative functions into itself. We shall further assume that the operator A on the cone K (in space C or one of the spaces L_p) is completely continuous and has a strong asymptotic derivative (7.31) with respect to a cone and a strong Fréchet derivative $A'(\theta)$ with respect to a cone at the point θ, where

$$A'(\theta)x(t) = \int_\Omega P(t, s)x(s)\,ds\,, \tag{7.36}$$

where

$$P(t, s)=k_u'(t, s, 0)\,. \tag{7.37}$$

The functions $P(t, s)$ and $Q(t, s)$ are non-negative by virtue of condition (7.28). We shall assume that these functions satisfy the additional restrictions by virtue of which the linear integral operators (7.31) and (7.36) have a unique normalized characteristic vector in the cone K. For this, for example, it suffices that the kernels $P(t, s)$ and $Q(t, s)$ satisfy the conditions of u_0-positiveness mentioned in subsection 7.1.1 of the present section. We shall denote by λ_0 and λ_∞ the greatest positive characteristic value (it corresponds to a non-negative characteristic function) of the operator $A'(\theta)$ and of the operator $A'(\infty)$ respectively.
Theorems 4.11 and 4.16 imply the next theorem.

Theorem 7.5. *The equation* (7.32) *has at least one non-negative non-identically zero solution for the above enumerated conditions if either*

$$\lambda_0<1<\lambda_\infty \tag{7.38}$$

or

$$\lambda_\infty<1<\lambda_0\,. \tag{7.39}$$

We give one more example. We consider the equation

$$x(t)=\int_\Omega k(t, s)f[s, x(s)]\,ds \tag{7.40}$$

with a continuous positive kernel $k(t, s)$ and a continuous and continuously differentiable function $f(s, u)$ satisfying the conditions

$$f(s, 0)\equiv 0\,,\quad f(s, u)\geqslant 0\,,\quad f(s, u)\geqslant au^{1+\varepsilon_0}-b(u\geqslant 0)\,, \tag{7.41}$$

where a, b, $\varepsilon_0>0$.

Theorem 7.6. *Let the spectrum of the kernel*

$$P(t, s)=k(t, s)f_u'(s, 0) \tag{7.42}$$

lie in a circle of radius $\varrho_0<1$. *Then the equation* (7.40) *has besides the zero solution at least one positive solution.*

Proof. Under the enumerated conditions, the operator

$$Ax(t)=\int_\Omega k(t, s)f[s, x(s)]\,ds$$

is completely continuous in C and leaves the cone $\tilde{K}$ (see subsection 7.1.6)

of non-negative functions satisfying the condition

$$\|x\| = \max_{t\in\Omega} x(t) \leqslant \varrho \min_{t\in\Omega} x(t) \tag{7.43}$$

invariant.

The operator A has a strong Fréchet derivative $A'(\theta)$ which is a linear integral operator with the kernel (7.42). By assumption, the spectrum of the operator $A'(\theta)$ lies in a circle of radius $\varrho_0 < 1$. Therefore, for the completion of the proof of the theorem, it suffices to show that the relation $Ax \overline{\leqslant} x$ is satisfied for $x(t)\in K$ and $\|x(t)\| \geqslant R$.

Assuming the contrary, we can find a sequence $x_n(t)\in\tilde{K}$ such that $\|x_n(t)\| \to\infty$ and

$$x_n(t) - Ax_n(t) \in \tilde{K} \subset K \qquad (n=1, 2, \ldots). \tag{7.44}$$

At the same time, by virtue of (7.41) and (7.43),

$$x_n(t) - Ax_n(t) = x_n(t) - \int_\Omega k(t, s) f(s, x_n(s)]\, ds \leqslant$$

$$\leqslant x_n(t) - \int_\Omega k(t, s)\{a|x_n(s)|^{1+\varepsilon_0} - b\}\, ds \leqslant$$

$$\leqslant \|x_n\| - \frac{a}{\varrho^{1+\varepsilon_0}} \|x_n\|^{1+\varepsilon_0} \int_\Omega k(t, s)\, ds + b \int_\Omega k(t, s)\, ds$$

and for sufficiently large numbers n the functions $x_n(t) - Ax_n(t)$ are negative and this contradicts (7.44).

This completes the proof of the theorem.

An analogous statement is not difficult to formulate for the equation (7.32).

7.1.9. *Characteristic functions*

We limit ourselves to one example.

Theorem 7.7. *Let the function $k(t, s, u)$ satisfy the inequality*

$$k(t, s, u) \geqslant P(t, s)u \qquad (t,\ s\in\Omega,\ u\geqslant 0). \tag{7.45}$$

Let the Uryson operator be completely continuous on the cone K of non-negative functions in the space E(C or L_p) in which the linear operator with the kernel $P(t, s)$ is continuous. Let

$$0 < m \leqslant P(t, s) \leqslant M < \infty \qquad (t,\ s\in\Omega). \tag{7.46}$$

Then the Uryson operator has a continuum of positive characteristic functions among which there are characteristic functions of an arbitrary norm.

Proof. For the proof it suffices to remark that, under the conditions of the theorem, the linear integral operator with the kernel $P(t, s)$ is a monotone minorant of the Uryson operator, and then to refer to Theorem 5.2. It follows from this theorem that, under the conditions of Theorem 7.7, the positive characteristic functions form a continuous branch of infinite length.

The condition (7.46) can be replaced by an arbitrary condition of u_0-positiveness (even u_0-boundedness below) of a linear integral operator with a kernel $P(t, s)$ (see subsection 7.1.1).

7.1.10. *Equation with concave non-linearities*

We continue the study of the integral Uryson operator

$$Ax(t) = \int_\Omega k[t, s, x(s)]\,ds\,.$$

In this section, it is assumed that $k(t, s, 0) \equiv$ and that the function $k(t, s, u)$ does not decrease as u increases, that is, for $u_1 < u_2$ the inequalities

$$k(t, s, u_1) \leqslant k(t, s, u_2) \qquad (t, s, \in \Omega)$$

are satisfied. This condition, obviously, guarantees the monotoneity of the operator with respect to the partial ordering defined by the cone of non-negative functions.

In the constructions of the present subsection the function (7.13) plays an essential role:

$$H(t, s, u) = \frac{1}{u}\,k(t, s, u) \qquad (u > 0)\,.$$

The following notation is used below:

$$u_0 = u_0(t) \equiv 1\,.$$

Let the function $H(t, s, u)$ decrease almost everywhere (see page 236) as u increases. Then, as it is easily seen, the inequality

$$k(t, s, u) > 0 \qquad (u \geqslant a) \tag{7.47}$$

is satisfied for almost all $s \in \Omega$ for an arbitrary number $t \in \Omega$ and an arbitrary number $a > 0$.

We first consider the Uryson operator in the space C of continuous functions on Ω. Let $x(t)$ be non-negative and not identically equal to zero.

Then a number $a>0$ can be found such that the set Ω_1 of those numbers $t\in\Omega$ in which $x(t)\geqslant a$ has a positive measure. It follows from this and from (7.47) that

$$Ax(t)=\int_{\Omega} k[t, s, x(s)]ds\geqslant\int_{\Omega_1} k[t, s, x(s)]ds>0 \qquad (t\in\Omega).$$

This means that the continuous function $Ax(t)$ on the compact set Ω is positive; consequently, it is bounded below and above by positive numbers α and β, that is,

$$\alpha u_0(t)\leqslant Ax(t)\leqslant\beta u_0(t) \qquad (t\in\Omega). \tag{7.48}$$

We show that the conditions for the decrease of the function $H(t, s, u)$ almost everywhere imply the u_0-concavity of the Uryson operator on the cone K of the non-negative functions of the space C (if, of course, the Uryson operator acts in the space C).

Let the function $x(t)$ satisfy the condition $\gamma u_0\leqslant x\leqslant\gamma_0 u_0$ $(\gamma>0)$, that is, $\gamma\leqslant x(t)\leqslant\gamma_0 (t\in\Omega)$. Let τ be some fixed number from (0, 1).

We choose a natural number n so that the inequality

$$\frac{\gamma_0-\tau\gamma}{n}<\frac{\gamma(1-\tau)}{2}$$

will be satisfied.

We divide the segment $[\tau\gamma, \gamma_0]$ into n equal subsegments by means of the points

$$\tau\gamma=\xi_0<\xi_1<\ \ldots\ <\xi_n=\gamma_0.$$

Let

$$\gamma\leqslant x\leqslant\gamma_9,\quad \xi_{i-1}\leqslant\tau x\leqslant\xi_i.$$

Then, obviously,

$$x=\tau x+(1-\tau)x\geqslant\xi_{i-1}+(1-\tau)\gamma>\xi_{i-1}+\frac{2}{n}(\gamma_0-\tau\gamma)=\xi_{i+1},$$

where for $i=n$ the number $\gamma_0=(\gamma_0-\tau\gamma)/n$ is denoted by ξ_{i+1}.

Therefore,

$$\begin{aligned}
&k(t, s,\tau x)-\tau k(t, s, x)=\\
&\quad=\tau x[H(t, s, \tau x)-H(t, s, x)]=\\
&\quad=\tau x\{[H(t, s, \tau x)-H(t, s, \xi_i)]+\\
&\qquad+[H(t, s, \xi_i)-H(t, s, \xi_{i+1})]+\\
&\qquad+[H(t, s, \xi_{i+1})-H(t, s, x)]\}\geqslant\\
&\quad\geqslant\tau x[H(t, s, \xi_i)-H(t, s, \xi_{i+1})],
\end{aligned}$$

and by virtue of the decrease almost everywhere of the function $H(t, s, u)$ the inequality

$$k(t, s, \tau x)-\tau k(t, s, x)>0$$

is satisfied for every t for arbitrary $x\in[\gamma, \gamma_0]$ for almost all $s\in\Omega$.
In particular, for a function $x(t)$ which satisfies the inequalities $\gamma\leqslant x(t)\leqslant\gamma_0$ for almost all $s\in\Omega$, the inequality

$$k[t, s, \tau x(s)]-\tau k[t, s, x(s)]>0$$

is satisfied. Integrating this inequality, we obtain

$$A[\tau x(t)]-\tau Ax(t)\geqslant 0 \qquad (t\in\Omega).$$

The difference in the left member is a continuous positive function. Therefore, a number μ can be found such that

$$A[\tau x(t)]-\tau Ax(t)>0 \qquad (t\in\Omega)$$

and by virtue of (7.48)

$$A[\tau x(t)]-\tau Ax(t)\geqslant\frac{\mu}{\beta}Ax(t) \qquad (t\in\Omega),$$

that is,

$$A(\tau x)\geqslant\tau\left(1+\frac{\mu}{\tau\beta}\right)Ax.$$

This completes the proof of the u_0-concavity of the operator A in the space C.
We shall say that $H(t, s, u)$ *decreases uniformly* as u increases if the difference $H(t, s, u_1)-H(t, s, u_2)$ for $u_1<u_2$ has a positive *infimum* with respect to $t, s\in\Omega$.
The uniform decrease of the function $H(t, s, u)$ implies the u_0-concavity in the space L_p of the Uryson operator on the cone of the non-negative functions (the proof is analogous to the case of the space C). In this connection, the condition guaranteeing the complete continuity of the Uryson operator on the cone of non-negative functions of the space $L_p(1<p<\infty)$ can be written in the form

$$\sup_t\int_\Omega|k_u'(t, s, 0)|^{p/(p-1)}ds<\infty.$$

By virtue of Theorem 6.2, the conditions indicated in this subsection for the concavity of an integral operator are conditions for the uniqueness of a positive solution for an integral equation

$$x(t)=\int_{\Omega} k[t, s, x(s)]\,ds\,.$$

Since a concave operator satisfies the requirements presented by monotone minorants, then under the conditions mentioned in the present subsection the Uryson operator has a continuum of positive characteristic functions. By virtue of theorems of chapter 6 these characteristic functions continuously and monotonely depend on a parameter.

In concluding this subsection, we note that the function $H(t, s, u)$ decreases as u increases if $k(t, s, u)(k(t, s, 0)\equiv 0,\ k(t, s, u)>0$ for $u>0)$ is concave with respect to the variable u.

7.1.11. *Equations with convex non-linearities*

For integral equations with a Uryson operator containing convex functions $k(t, s, u)$ with respect to u, sufficiently general theorems for the uniqueness of a positive solution are unknown. A general method for the proof of such theorems is also unknown.

In some particular cases, we can succeed in using the principle of uniqueness mentioned in chapter 6 (page 207) for equations with convex operators. The example mentioned below is of interest for the basic method of the proof. The reader can without difficulty generalize it for equations with Uryson operators. We consider the equation

$$x(t)=\int_{\Omega} k(t, s)f[x(s)]\,ds \tag{7.49}$$

under the following conditions. The function $k(t, s)$ is continuous with respect to the collection of variables and

$$0<m\leqslant k(t, s)\leqslant M<\infty \qquad (t, s\in\Omega)\,, \tag{7.50}$$

the function $f(u)$ $(u\geqslant 0)$ is continuous, non-negative, monotone and convex, $f(0)=0$ and $f'(0)=1$. For the sake of simplicity, we shall assume that mes $\Omega=1$.

Under the enumerated restrictions, the operator

$$Ax(t)=\int_{\Omega} k(t, s)f[x(s)]\,ds \tag{7.51}$$

is completely continuous in the space C and leaves the cone K of non-negative functions invariant. Moreover, the operator A leaves the cone $\tilde{K}$ of non-negative functions $x(t)$ such that

$$\min x(t) \geqslant \frac{m}{M} \max x(t) = \frac{m}{M} \|x(t)\|$$

invariant.

The operator A is a u_0-convex operator on the cone $\tilde{K}$ (the proof is developed by the same arguments by means of which the conditions for the u_0-concavity of the Uryson operator were found in the preceding chapter). The Fréchet derivative $A'(\theta)$ of the operator (7.51) is defined by the equality

$$A'(\theta)h(t) = \int_\Omega k(t, s)h(s)\,ds\,,$$

and by virtue of (7.50) the spectrum of the operator $A'(\theta)$ lies in a circle of radius M. We shall assume that

$$M < 1\,. \tag{7.52}$$

In the sequel, it will be convenient for us to write the function $f(u)$ in the form

$$f(u) = u[1 + \varphi(u)] \qquad (u \geqslant 0)\,.$$

We shall assume that $\varphi(u)$ is monotone and $\varphi(u) \to \infty$ as $u \to \infty$. Then the inequality

$$Ax(t) \geqslant m \int_\Omega x(s)\{1 + \varphi[x(s)]\}\,ds \geqslant \frac{m^2}{M} \|x\|\, \varphi\left(\frac{m}{M} \|x\|\right)$$

will be satisfied for the functions $x(t) \in \tilde{K}$ from which it follows that for the functions $x(t) \in \tilde{K}$ with a large norm the relation $Ax \leqslant x$ is satisfied. The existence under condition (7.52) of at least one solution for the equation (7.49) follows from the theorem about fixed points of operators which expand a cone.

We assume that the equation (7.49) has two positive solutions $x_1(t)$ and $x_2(t)$. It follows from the uniqueness principle for convex operators that both of the differences $x_1(t) - x_2(t)$ and $x_2(t) - x_1(t)$ are not strictly positive functions. Without loss of generality, it can be assumed that the difference

$$y(t) = x_1(t) - x_2(t)$$

has the following property: numbers t_0 and t_1 can be found such that

$$y(t_0) = \max_{t \in \Omega} y(t) = \|y(t)\|\,, \qquad y(t_1) \leqslant 0\,. \tag{7.53}$$

For every function $x_i(t)$ $(i = 1, 2)$ the estimates

$$\|x_i(t)\| = \max_{t\in\Omega} \int_\Omega k(t, s)f[x_i(s)]ds \leqslant M\|x_i\|[1+\varphi(\|x_i\|)]$$

$$\min x_i(t) = \min_{t\in\Omega} \int_\Omega k(t, s)f[x_i(s)]ds \geqslant$$

$$\geqslant m \min_{t\in\Omega} |x_i(t)|\{1+\varphi[\min_{t\in\Omega} x_i(t)]\} \geqslant$$

$$\geqslant \min_{t\in\Omega} x_i(t)m\left[1+\varphi\left(\frac{m}{M}\|x_i\|\right)\right]$$

are obvious from which it follows that

$$\varphi^{-1}\left(\frac{1}{M}-1\right) \leqslant \|x_1\|,\ \|x_2\| \leqslant \frac{M}{m}\varphi^{-1}\left(\frac{1}{m}-1\right), \tag{7.54}$$

where $\varphi^{-1}(u)$ denotes the inverse function to $\varphi(u)$. From the equalities

$$x_1(t) = \int_\Omega k(t, s)f[x_1(s)]ds$$

and

$$x_2(t) = \int_\Omega k(t, s)f[x_2(s)]ds$$

it follows that

$$y(t) = \int_\Omega k(t, s)f'[x(s)]y(s)ds, \tag{7.55}$$

where the function $\tilde{x}(s)$ assumes values which are between the values $x_1(s)$ and $x_2(s)$, By virtue of inequality (7.54) (and since $x_i(t)\in\tilde{K}$, $i=1, 2$) the function $\tilde{x}(s)$ satisfies the inequalities

$$\frac{m}{M}\varphi^{-1}\left(\frac{1}{M}-1\right) \leqslant \tilde{x}(s) \leqslant \frac{M}{m}\varphi^{-1}\left(\frac{1}{m}-1\right).$$

Therefore

$$a_1 = mf'\left[\frac{m}{M}\varphi^{-1}\left(\frac{1}{M}-1\right)\right] \leqslant k(t, s)f'[\tilde{x}(s)] \leqslant$$

$$\leqslant Mf'\left[\frac{M}{m}\varphi^{-1}\left(\frac{1}{m}-1\right)\right] = a_2 \qquad (t, s\in\Omega).$$

We rewrite the equality (7.55) in the form

$$y(t) - \frac{a_1+a_2}{2}\int_\Omega y(s)ds = \int_\Omega \left[k(t, s)f'[\tilde{x}(s)] - \frac{a_1+a_2}{2}\right]y(s)ds. \tag{7.56}$$

It follows from (7.53) that

$$\|y(t)-b\| \geqslant \tfrac{1}{2}\|y(t)\|$$

for an arbitrary constant b. In particular,

$$\|y(t) - \frac{a_1+a_2}{2}\int_\Omega y(s)\,ds\| \geqslant \tfrac{1}{2}\|y(t)\| .$$

Consequently, it follows from (7.56) that

$$\tfrac{1}{2}\|y(t)\| \leqslant \sup_{t,s\in\Omega}\left|k(t,s)f'[\tilde{x}(s)] - \frac{a_1+a_2}{2}\right| \|y\| \leqslant \frac{a_2-a_1}{2}\|y(t)\| ,$$

that is,

$$Mf'\left[\frac{M}{m}\varphi^{-1}\left(\frac{1}{m}-1\right)\right] - mf'\left[\frac{m}{M}\varphi^{-1}\left(\frac{1}{M}-1\right)\right] \geqslant 1 .$$

If the last inequality does not hold, then the equation (7.49) has a unique positive solution. We have thus proved the following theorem.

Theorem 7.8. *The equation* (7.49) *has a unique solution if*

$$Mf'\left[\frac{M}{m}\varphi^{-1}\left(\frac{1}{m}-1\right)\right] - mf'\left[\frac{m}{M}\varphi^{-1}\left(\frac{1}{M}-1\right)\right] < 1. \tag{7.57}$$

We remark that condition (7.57) evidently is satisfied if m is sufficiently

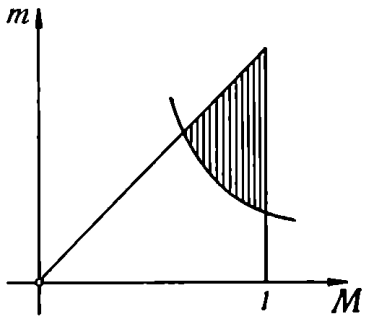


Fig. 4.

near to M since the left member of the inequality approaches zero as $m \to M$.

The condition (7.57) assumes a more visible form for particular classes of functions. For example, let $f(x)=x(1+\alpha x^r)$. Then

$$\varphi(x)=\alpha x^r , \quad \varphi^{-1}(u)=\left(\frac{u}{\alpha}\right)^{1/r} , \quad f'(x)=1+\alpha(r+1)x^r .$$

Condition (7.57) is written in the form

$$\left(\frac{M^{r+1}}{m^{r+1}}+1\right)(1-m)<1+\left(\frac{m^{r+1}}{M^{r+1}}+1\right)(1-M). \tag{7.58}$$

For the admissible values $\{m, M\}$ in the $\{m, M\}$ plane, we obtain the domain shaded on Fig. 4. Without essential changes, the arguments mentioned can be generalized to the case when the kernel $k(t, s)$ satisfies the inequalities

$$\alpha_0 u(t)\leqslant k(t, s)\leqslant \beta_0 u(t)\ ,$$

where $\alpha_0, \beta_0>0$ and $u(t)$ is some non-negative function.

7.1.12. *Remarks*

For lack of space and in order to avoid unwieldy formulations, we have not presented applications to systems of integral equations. These applications can be obtained following the same plan which was applied above for the consideration of a single equation. However, in this connection, the integral operators must be considered in spaces of vector functions. Several corollaries for integral equations not requiring special constructions are completely omitted by us. For example, Theorem 6.9 concerning the points of bifurcation applies without any changes to differentiable integral operators. Under the conditions of subsection 7.1.10, a positive solution of the Uryson equation can be obtained as the uniform limit of the successive approximations

$$x_n(t)=\int_\Omega k[t, s, x_{n-1}(s)]\,ds \qquad (n=1, 2, \ldots)\ .$$

We further note that some particular classes of integral operators are considered in the following sections.

7.2. First boundary value problem for elliptic equations of the second order with non-linearities

7.2.1. *Elliptic operator*

Let Ω be a bounded open domain of an N-dimensional space. The boundary Γ of the domain Ω is assumed to be sufficiently smooth.

The differential expression

$$Lu(t) = -\sum_{i,j=1}^{N} a_{ij}(t, \ldots, t_N) \frac{\partial^2 u}{\partial t_i \partial t_j} + \\ + \sum_{i=1}^{N} a_i(t_1, \ldots, t_N) \frac{\partial u}{\partial t_i} + a(t_1, \ldots, t_N)u \tag{7.59}$$

is considered where $t=\{t_1, \ldots, t_N\} \in \bar{\Omega}$, $\bar{\Omega}=\Omega+\Gamma$. In the sequel, it is assumed that

$$\sum_{i,j=1}^{N} a_{ij}(t)\xi_i\xi_j \geqslant \gamma \sum_{i=1}^{N} \xi_i^2 \qquad (\gamma>0) \tag{7.60}$$

for arbitrary $\xi_1, \ldots, \xi_N$ and the function $a(t_1, \ldots, t_N)$ is non-negative. For the sake of simplicity, we shall assume that the coefficients of the differential expression (7.59) are sufficiently smooth (it suffices to assume that the coefficients $a_{ij}(t_1, \ldots, t_N)$ have first derivatives and that these first derivatives, and also the functions $a_i(t_1, \ldots, t_N)$ and functions $a(t_1, \ldots, t_N)$, satisfy some Hölder condition in a closed domain).

The *first boundary value problem* is the problem about the determination of a solution of the differential equation

$$Lu(t)=v(t)\,, \tag{7.61}$$

which satisfies the zero boundary condition

$$u(t)_{t\in\Gamma}=0\,. \tag{7.62}$$

By a regular solution of the first boundary value problem is understood a function $u(t)$ which is continuously differentiable in Ω and continuous on $\bar{\Omega}$ and satisfies the equation (7.61) in Ω and becomes zero on the boundary Γ of the domain Ω.

If the function $v(t)$ is sufficiently smooth, then, as it is known, a regular solution exists and is representable in the form

$$u(t)=Av(t)=\int_{\Omega} G(t, s)v(s)\,ds\,, \tag{7.63}$$

where the kernel $G(t, s)$ is called the *Green function of a first boundary value problem.*

The Green function $G(t, s)$ is continuous with respect to the collection of variables for $t\in\bar{\Omega}$, $s\in\Omega$ and $s\neq t$. For $N>2$ the estimate

$$0 \leqslant G(t, s) \leqslant \frac{k_0}{|t-s|^{N-2}} \tag{7.64}$$

holds, and for

$$0 \leqslant G(t, s) \leqslant k_0 |\ln| t-s| \,|\,, \tag{7.65}$$

where the distance between the points t and s is denoted by $|t-s|$.
It follows from the estimates (7.64) and (7.65) and from the theorems of S. L. Sobolev about operators of potential type that the operator (7.63) acts from every $L_p (1<p<\infty)$, where

$$p > \frac{N}{2}, \tag{7.66}$$

into the space C of continuous functions on $\bar{\Omega}$ and is a completely continuous operator.
If the function $v(t)$ does not have the properties of sufficient smoothness, then the first boundary value problem cannot have regular solutions. In this case, the concept of a solution of the boundary value problem is extended. One of the most natural methods for such an extension consists in the fact that the solution is defined by the formula (7.63). It then follows from the arguments given above that the solution of the first boundary value problem exists for an arbitrary function $v(t) \in L_p$.
It follows from the results of A. I. Koshelev [1] that the solutions of the first boundary value problem are continuously differentiable for $v(t) \in L_p$ if $p > N$.
In particular, the solutions $Av(t)$ are continuously differentiable for continuous $v(t)$.
A stronger statement also follows from the results of Koshelev. The operators

$$B_i v(t) = \frac{\partial}{\partial t_i} Av(t) \qquad (i = 1, \ldots, N) \tag{7.67}$$

are completely continuous operators acting from the spaces L_p, where $p > N$, into the space C of continuous functions on $\bar{\Omega}$. In particular, the operators (7.67) are completely continuous in C.

7.2.2. *Integral inequality for the Green function*

As it is known, the Green function of the first boundary value problem is non-negative (this follows from the principle of the maximum; see Miranda [1]). Therefore, the operator (7.63) leaves the cone of non-negative functions invariant.
It follows from the non-negativeness of the Green function that

$$\int_{\Omega_1} G(t, s)\,ds \leqslant \int_{\Omega_2} G(t, s)\,ds \qquad (t \in \Omega),$$

if $\Omega_1 \subset \Omega_2$. It turns out that the reverse inequality, in a definite sense, can be established.

Lemma 7.2. *Let Ω_0 be a closed subdomain of the domain Ω. Then a number $\varepsilon = \varepsilon(\Omega_0) > 0$ can be found such that*

$$\varepsilon \int_{\Omega} G(t, s)\,ds \leqslant \int_{\Omega_0} G(t, s)\,ds \qquad (t \in \bar{\Omega}). \tag{7.68}$$

Proof. We denote by $\varphi_0(t)$ $(t \in \bar{\Omega})$ the function identically equal to 1, and by $\varphi_1(t)$ a continuously differentiable function equal to zero on $\bar{\Omega} \setminus \Omega$. and equal to 1 on some domain $\Omega_1 \subset \Omega_0$ and assuming values from [0, 1] for the remaining points.

The functions $A\varphi_0(t)$ and $A\varphi_1(t)$ will be positive for $t \in \Omega$ and equal to zero for $t \in \Gamma$. They are regular continuously differentiable solutions of the problem (7.61)–(7.62) for $v(t) = \varphi_0(t)$ and $v(t) = \varphi_1(t)$ respectively.

It follows from the Jiro-Oleinik estimates (see, for example Miranda [1]) that the normal derivatives of the functions $A\varphi_0(t)$ and $A\varphi_1(t)$ are positive at all points $t \in \Gamma$.

Therefore, a number $\varepsilon_0 > 0$ can be found such that the normal derivative of the functions

$$\psi_\varepsilon(t) = A\varphi_1(t) - \varepsilon A\varphi_0(t)$$

for $0 < \varepsilon \leqslant \varepsilon_0$ will be positive at all the points $t \in \Gamma$. For these values of ε the functions $\psi_\varepsilon(t)$ will assume non-negative values in some bounded region since the Green function $G(t, s)$ is bounded below by a positive constant when the distance from the points t and s to Γ is greater than some positive number, then the functions $A\varphi_0(t)$ and $A\varphi_1(t)$ are strictly positive at all interior points of the domain Ω. Therefore, the functions $\psi_\varepsilon(t)$ are non-negative for the entire closed domain $\bar{\Omega}$ for sufficiently small ε, that is,

$$\varepsilon \int_{\Omega} G(t, s)\,ds \leqslant \int_{\Omega} G(t, s)\varphi_1(s)\,ds \qquad (t \in \bar{\Omega}).$$

But it follows from the non-negativeness of the Green function that

$$\int_{\Omega} G(t, s)\varphi_1(s)\,ds \leqslant \int_{\Omega_0} G(t, s)\,ds \qquad (t \in \bar{\Omega}).$$

This completes the proof of the lemma.

We introduce the notation

$$u_0(t) = \int_\Omega G(t, s)\,ds\,. \tag{7.69}$$

Lemmas 7.1. and 7.2. imply the following theorem.

Theorem 7.9. *The linear integral operator* (7.63), *for which the Green function* $G(t, s)$ *of the problem* (7.61)–(7.62) *is a kernel, is* u_0*-bounded.*

Various statements about a non-negative characteristic function of the operator (7.63) or, what is the same, about characteristic functions of the problem

$$\lambda Lu(t) = u(t), \quad u(t)|_{t\in\Gamma} = 0 \tag{7.70}$$

follow from this theorem and from the results of the second chapter. We formulate a part of these statements.

Theorem 7.10. *Problem* (7.70) *has a unique normalized non-negative characteristic function to which there corresponds the positive characteristic value* λ_0. *The characteristic value* λ_0 *is simple. The remaining characteristic values are in absolute value strictly less than* λ_0.

7.2.3. *Non-homogeneous linear equations*
We consider the problem

$$Lu(t) = \alpha u(t)\,, \quad u(t)_{t\in\Gamma} = 0\,. \tag{7.71}$$

We shall denote by λ_0 a characteristic value of the problem (7.70) to which there corresponds a non-negative characteristic function.
Problem (7.71) is equivalent to the equation

$$u(t) = \alpha A u(t) + f_1(t)\,, \tag{7.72}$$

where A is the operator (7.63), $f_1(t) = Af(t)$.
Let $f(t)$ be non-negative and not identically equal to zero; then the function $f_1(t)$ also is non-negative and not identically equal to zero. It then follows from Theorems 7.9 and 2.16 that problem (7.71) has a unique non-negative solution if $\alpha\lambda_0 < 1$; problem (7.71) does not have non-negative solutions if $\alpha\lambda_0 \geqslant 1$.

7.2.4. *Existence of a non-negative solution*
We turn to the study of the non-linear equation

$$Lu(t)=f\left(t_1, \ldots, t_N, u, \frac{\partial u}{\partial t_1}, \ldots, \frac{\partial u}{\partial t_N}\right) \tag{7.73}$$

with a zero boundary condition (7.62). We shall assume that the function

$$f(t, u, v)\equiv f(t, \ldots, t_N, u, v_1, \ldots, v_N)$$

is continuous with respect to the collection of variables $t=\{t_1, \ldots, t_N\}$, u and $v=\{v_1, \ldots, v_N\}$ in the domain

$$\tilde{\Omega}: t\in\Omega\,, \quad u\geqslant 0\,, \quad -\infty<v_1, \ldots, v_N<\infty$$

and assumes non-negative values in the domain.

We shall be interested in the question about the existence of non-negative solutions of the problem (7.73)–(7.62).

Naturally such solutions do not always exist. For example, let

$$f(t, u, v)\geqslant \alpha u \qquad (\{t, u, v\}\in\tilde{\Omega}) \tag{7.74}$$

and

$$\alpha\lambda_0\geqslant 1\,, \tag{7.75}$$

where λ_0 is the greatest characteristic value of problem (7.70). In this connection, we assume that problem (7.73)–(7.62) has a non-negative solution $u^*(t)$. It then follows from the arguments of the preceding subsection that the function $u^*(t)$ satisfies the identity

$$f\left[t_1, \ldots, t_N, \quad u^*(t), \quad \frac{\partial u^*(t)}{\partial t_1}, \ldots, \frac{\partial u^*(t)}{\partial t_N}\right]\equiv 0\,.$$

Consequently, $u^*(t)$ is a solution of the problem

$$Lu=0\,, \quad u(t)|_{t\in\Gamma}=0\,.$$

This means that $u^*(t)\equiv 0$. Thus, if conditions (7.74) and (7.75) are satisfied, then the problem (7.73)–(7.62) does not have non-negative solutions which are different from zero.

We introduce for consideration the operator

$$Fw(t)=f[t_1, \ldots, t_N, \quad Aw(t), \quad B_1w(t), \ldots, B_Nw(t)]\,, \tag{7.76}$$

where $A, B_1, \ldots, B_N$ are the operators (7.63) and (7.67), respectively. As was explained in subsection 7.2.1, the operators $A, B_1, \ldots B_N$ are completely continuous in the space C. Therefore, the operator F is completely continuous in C.

It follows from the non-negativeness of the function $f(t, u, v)$ in the domain $\tilde{\Omega}$ that the operator F transforms the cone K of non-negative con-

tinuous functions into itself. This fact allows the application of methods developed in the preceding chapter to the investigation of non-negative solutions of the equations

$$\boldsymbol{F}w(t)=w(t) \tag{7.77}$$

and

$$\boldsymbol{F}w(t)=\lambda w(t)\,, \tag{7.78}$$

where $\lambda>0$.

It is not difficult to see that equation (7.77) is equivalent to the boundary value problem (7.73)–(7.62). In fact, if $u^*(t)$ is the solution of this boundary value problem, then the function

$$w^*(t)=\boldsymbol{L}u^*(t) \tag{7.79}$$

will be a solution of the equation

$$\boldsymbol{F}w(t)=f[t_1, \ldots, t_N, \quad \boldsymbol{A}w(t), \quad \boldsymbol{B}_1 w(t), \ldots, \boldsymbol{B}_N w(t)]$$

or, what is the same, equation (7.77). However, if $w^*(t)$ is a solution of the equation (7.77), then the function

$$u^*(t)=\boldsymbol{A}w^*(t) \tag{7.80}$$

will be a solution of problem (7.73)–(7.62).

Analogously, the formulas (7.79) and (7.80) establish a one-to-one correspondence between the solutions of the equation (7.78) and the solutions of the boundary value problem

$$\lambda \boldsymbol{L}u=f\left(t_1, \ldots, t_N, \quad u, \quad \frac{\partial u}{\partial t_1}, \ldots, \frac{\partial u}{\partial t_N}\right), \qquad u(t)|_{t\in\Gamma}=0\,. \tag{7.81}$$

We must still note that to the non-negative solutions of equation (7.77) or (7.78) there corresponds a non-negative solution $u^*(t)$ of a corresponding boundary value problem—this follows from the positiveness of the operator $\boldsymbol{A}$. However, if $u^*(t)$ is a non-negative solution of a boundary value problem, then the function

$$f\left[t_1, \ldots, t_N, \quad u^*(t), \quad \frac{\partial u^*(t)}{\partial t_1}, \ldots, \frac{\partial u^*(t)}{\partial t_N}\right]$$

is non-negative and, therefore, the solution (7.79) of the corresponding equation (7.77) or (7.78) is non-negative.

We again assume that condition (7.74), in which α is some positive

number, is satisfied. Then the inequalities

$$\boldsymbol{F}x(t) \geqslant \alpha A x(t) \qquad (t \in \bar{\Omega})$$

will be satisfied for every non-negative and continuous function $x(t)$ on $\bar{\Omega}$. These inequalities mean that the u_0-positive (see 2) linear operator αA is a monotone minorant of the operator $\boldsymbol{F}$. It follows from Theorem 5.7 that equation (7.78) has a continuum of non-negative solutions $w^*(t)$ not equal to zero which correspond to certain positive values of λ. It follows from the arguments presented in the beginning of this subsection that these values satisfy the inequality

$$\lambda > \alpha\lambda_0 .$$

Turning from equation (7.78) to the boundary value problem (7.81), we arrive at the conclusion that *condition* (7.74) *is sufficient for the non-linear boundary value problem* (7.81) *to have a continuum of distinct non-negative solutions each of which corresponds to some positive* λ.

We return to the study of the non-linear equation (7.73).

We first assume that the function $f(t, u, v)$ satisfies the condition

$$0 \leqslant f(t, u, v) \leqslant a + bu \qquad (\{t, u, v\} \in \bar{\Omega}) , \tag{7.82}$$

where $a, b \geqslant 0$. Then the operator (7.76) satisfies the inequalities

$$0 \leqslant \boldsymbol{F}w(t) \leqslant a + bAw(t) \qquad (t \in \bar{\Omega},\ w(t) \geqslant 0) . \tag{7.83}$$

If the additional condition

$$b\lambda_0 < 1 \tag{7.84}$$

is satisfied, where λ_0 is the greatest characteristic value of the operator A, *then the equation* $w = \boldsymbol{F}w$ *has at least one solution in the cone* K *of non-negative functions*. This statement is a direct corollary to Theorem 4.10. We have thus proved the following theorem.

Theorem 7.11. *The boundary value problem* (7.73)–(7.62) *has at least one non-negative solution if conditions* (7.82) *and* (7.84) *are satisfied.*

7.2.5. *Existence of a non-negative solution which is not identically equal to zero*

If

$$f(t_1, \ldots, t_N, 0, 0, \ldots, 0) \equiv 0 \qquad (\{t_1, \ldots, t_N\} \in \bar{\Omega}) , \tag{7.85}$$

then the boundary value problem (7.73)–(7.62) has the trivial solution. We find some simple conditions for which this problem has non-trivial

non-negative solutions.
We shall assume that $f(t_1, \ldots, t_N, u, v_1, \ldots, v_N)$ is continuously differentiable with respect to the variables $u, v_1, \ldots, v_N$ in a neighborhood of the zero point. It follows from the non-negativeness of the function $f(t, u, v)$ in the domain $\bar{\Omega}$ that

$$f_{v_1}'(t_1, \ldots, t_N, 0, 0, \ldots, 0) \equiv \ldots \equiv f_{v_N}'(t_1, \ldots, t_N, 0, 0, \ldots, 0) \equiv 0 .$$

The function

$$g(t) = g(t_1, \ldots, t_N) = f_u'(t_1, \ldots, t_N, 0, 0, \ldots, 0)$$

is non-negative. The operator (7.76) will have a strong Fréchet derivative at the point θ, where

$$\boldsymbol{F}'(\theta)x(t) = g(t)\boldsymbol{A}x(t) . \tag{7.86}$$

The proof is obvious.
We assume that the function $g(t)$ is different from zero. Let $u_0(t)$ be the function (7.69).

Lemma 7.3. *The operator* (7.86) *is* v_0*-positive where* $v_0(t) = g(t)u_0(t)$.

Proof. In the proof of Lemma 7.1, it was shown that the validity of the inequalities

$$\alpha u_0(t) \leqslant \boldsymbol{A}x(t) \leqslant \beta u_0(t) \qquad (t \in \bar{\Omega}) ,$$

where α and β are positive for every non-negative and non-zero function $x(t)$, follows from the inequalities (7.68). Then the inequalities

$$\alpha v_0(t) \leqslant \boldsymbol{F}'(\theta)x(t) \leqslant \beta v_0(t) \qquad (t \in \bar{\Omega})$$

are satisfied.
This completes the proof of the lemma.

By virtue of Lemma 7.3, the linear operator $\boldsymbol{F}'(\theta)$ has a unique normalized characteristic non-negative function to which a positive characteristic value $\Lambda(\theta)$ corresponds.

Theorem 7.12. *Let the conditions* (7.82) *and* (7.84) *be satisfied. Let* $\Lambda(\theta) > 1$. *Then the boundary value problem* (7.73)–(7.62) *has besides the trivial solution at least one more characteristic solution.*

For the proof, it suffices to deduce the existence of a non-trivial solution for the equation (7.77) from the theorems about operators compressing

the cone, and, after this, construct the solution sought according to formula (7.80).

Lemma 7.4. *Let the function $f(t, u, v)$ satisfy the condition* (7.82) *of sublinearity and let*

$$\lim_{u\to\infty}\sup_{t\in\bar{\Omega},\ -\infty<v_i<\infty}\left|\frac{1}{u}f(t, u, v)-h(t)\right| = 0\,, \tag{7.87}$$

where $h(t)=h(t_1, \ldots, t_N)$ is a continuous non-negative function.
Then the operator F has a strong asymptotic derivative $F'(\infty)$ with respect to the cone K where

$$F'(\infty)x(t)=h(t)Ax(t)\,. \tag{7.88}$$

The proof is essentially obvious. Assuming the contrary, we can find a sequence of non-negative functions $x_n(t)$ such that

$$\lim_{n\to\infty}\|x_n(t)\|=\infty$$

and

$$\|Fx_n(t)-h(t)Ax_n(t)\|\geqslant\beta_0\|x_n(t)\| \qquad (n=1, 2, \ldots)\,, \tag{7.89}$$

where β_0 is some positive number.
From (7.87) follows the existence of a number $\gamma_0>0$ such that the inequality

$$|f(t, u, v)-h(t)u|<\tfrac{1}{2}\beta_0 u \tag{7.90}$$

is satisfied for $u\geqslant\gamma_0$. We define the functions $z_n(t)$ by the equalities

$$z_n(t) = \begin{cases} Ax_n(t), & \text{if} \quad Ax_n(t)\geqslant\gamma_0\,, \\ \gamma_0 & \text{if} \quad Ax_n(t)<\gamma_0\,. \end{cases}$$

It follows from the conditions (7.90) and (7.82) that

$$\begin{aligned}
&\max_{t\in\bar{\Omega}}|Fx_n(t)-h(t)Ax_n(t)|\leqslant \\
&\quad\leqslant\max_{t\in\bar{\Omega}}|f[t, z_n(t), B_1x_n(t), \ldots, B_Nx_n(t)]-h(t)z_n(t)|+ \\
&\qquad\qquad\max_{t\in\bar{\Omega},\ 0\leqslant u\leqslant\gamma_0}|f[t, u, B, x_n(t), \ldots, B_Nx_n(t)]|+\gamma_0\max_{t\in\bar{\Omega}}h(t)\leqslant \\
&\quad\leqslant\tfrac{1}{2}\beta_0\|z_n(t)\|+a+b\gamma_0+\gamma_0\|h(t)\|\,,
\end{aligned}$$

that is,

$$\|Fx_n(t)-h(t)Ax_n(t)\|\leqslant M+\tfrac{1}{2}\beta_0\|x_n(t)\|\,,$$

which contradicts (7.89).
This completes the proof of the lemma.

Analogous to how Lemma 7.3 was proved, it can be shown that the operator (7.88) is v_1-bounded where $v_1(t)=h(t)u_0(t)$. Therefore, the operator (7.88) has a unique non-negative characteristic function to which there corresponds the characteristic value $\Lambda(\infty)$.
The greatest characteristic value of the linear operator (7.86) (for the case when the function $g(t)$ is not identically equal to zero) was denoted above by $\Lambda(\theta)$. In the case when

$$f_u'(t_1, \ldots, t_N, 0, 0, \ldots, 0)\equiv 0\,,$$

the operator $\boldsymbol{F}$ has at the point θ a strong Fréchet derivative which equals zero; in this case, we shall assume that $\Lambda(\theta)=0$.

Theorem 7.13. *Let*

$$\Lambda(\theta)<1<\Lambda(\infty)\,.$$

Then the non-linear boundary value problem (7.73)–(7.62) *has besides the trivial zero solution at least one more non-negative solution.*

Proof. It suffices for the proof to deduce the existence of a non-trivial non-negative solution $w^*(t)$ for equation (7.77) from the theorems concerning operators expanding the cone, and thus construct the solution of the boundary value problem with respect to formula (7.80).
For lack of space, we limit ourselves to the examples cited.

7.2.6. *Uniqueness of a positive solution*
We consider here the simpler problem

$$\boldsymbol{L}u=f(t, u)\,, \qquad u(t)|_{t\in\Gamma}=0\,. \tag{7.91}$$

We replace this problem by the equivalent integral equation

$$u(t) = \int_\Omega G(t, s)f[s, u(s)]\,ds\,, \tag{7.92}$$

where $G(t, s)$ is the Green function of the differential operator $\boldsymbol{L}$ for zero boundary conditions.
We shall assume that the function $f(t, u)$ is non-negative and is concave with respect to the variable u in the following generalized sense: for $u>0$ and arbitrary $\tau\in(0, 1)$

$$f(t, \tau, u)-\tau f(t, u)>0 \qquad (t\in\Omega)\,. \tag{7.93}$$

Lemma 7.5.* *If condition* (7.93) *is satisfied, then the operator*

$$Fx(t) = \int_{\Omega} G(t, s) f([s, x(s)])\, ds \tag{7.94}$$

is u_0-concave where $u_0(t)$ is the function (7.69).

Proof. If $u(t)$ is non-negative, then $f[t, u(t)]$ is non-negative where $f[t, u(t)]$ is positive (by virtue of (7.93)) at those points in which $u(t)$ is positive. It follows from Theorem 7.9 that condition (6.1) is satisfied.
If $u(t)$ is non-negative and not identically equal to zero, then for an arbitrary $\tau_0 \in (0, 1)$ the inequality

$$f[t, \tau_0 u(t)] - \tau_0 f[t, u(t)] > 0 \tag{7.95}$$

is satisfied at those points $t \in \Omega$ in which $u(t) > 0$. Again, it follows from Theorem 7.9 that

$$\int_{\Omega} G(t, s)\{f[s, \tau_0 u(s)] - \tau_0 f[s, u(s)]\}\, ds \geqslant \alpha u_0(t) \qquad (t \in \bar{\Omega}),$$

where $\alpha > 0$. But

$$\int_{\Omega} G(t, s) f[s, u(s)]\, ds \leqslant \beta_1 u_0(t) \qquad (t \in \bar{\Omega}).$$

Therefore,

$$\int_{\Omega} G(t, s) f[s, \tau_0 u(s)]\, ds \geqslant \tau_0 \left(1 + \frac{\alpha}{\beta_1 \tau_0}\right) \int_{\Omega} G(t, s) f[s, u(s)]\, ds \quad (t \in \bar{\Omega}).$$

The last inequality coincides with condition (6.5) for u_0-concavity.
This completes the proof of the lemma.

It follows from Lemma 7.5 and Theorem 6.3 that for the uniqueness of a non-trivial non-negative solution for equation (7.92) it suffices that operator (7.94) be monotone. That the function $f(t, u)$ be non-decreasing with respect to the variable u is sufficient for the monotoneity of operator (7.94).

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* The statement of the lemma is relative to arbitrary non-linear integral operators of the form (7.92) in which the kernel $G(t, s)$ defines the integral linear operator A having the property that for every non-negative and not identically equal to zero function $x(t)$ the inequalities

$$\alpha u_0(t) \leqslant Ax(t) \leqslant \beta u_0(t)$$

are satisfied, where α and β are positive and $u_0(t)$ is some fixed function.

Equation (7.92) is equivalent to problem (7.91). Therefore, we have proved the following theorem.

Theorem 7.14. *If $f(t, u)$ does not decrease with respect to the variable u and a satisfies the condition* (7.93), *then the boundary value problem* (7.91) *does not have more than one non-negative solution which is not identically equal to zero.*

This statement can be supplemented by the theorem about convergence of successive approximations to a unique non-negative solution.

Theorem 7.15. *Let $f(t, u)$ not decrease with respect to the variable u and let it satisfy the condition* (7.93).
Then the boundary value problem

$$\lambda Lu = f(t, u), \quad u(t)|_{t\in\Gamma} = 0 \tag{7.96}$$

has non-zero non-negative solutions for the values λ, filling some interval. These solutions $u(t; \lambda)$ depend monotonely and continuously on λ.

Theorem 7.15 contains corollaries to part of the results studied in section 6.2. The reader can without difficulty formulate corollaries to the other theorems indicated there.

7.2.7. *About quasi-linear equations*

The general theory of positive solutions of non-linear operator equations can be useful in the study of quasi-linear equations, for example, of the form

$$L(u)u \equiv -\sum_{i,j=1}^{N} a_{ij}(t, u)\frac{\partial^2 u}{\partial t_i \partial t_j} + \sum_{i=1}^{N} a_i(t, u)\frac{\partial u}{\partial t_i} + a(t, u)u = $$

$$= f\left(t, u, \frac{\partial u}{\partial t_1}, \ldots, \frac{\partial u}{\partial t_N}\right). \tag{7.97}$$

We again assume that the first boundary value problem is being considered, that is, solutions are sought which satisfy the zero boundary condition (7.62).

With respect to the coefficients, we shall assume that they are sufficiently smooth and satisfy conditions for which the problem

$$L(u)u = f(t), \quad u(t)|_{t\in\Gamma} = 0 \tag{7.98}$$

has a unique solution for an arbitrary function $f(t)$ from some functional

space (for example, for an arbitrary continuous $f(t)$). This solution $u(t)$ defines the non-linear operator L^{-1} by the equality $L^{-1}f=u$.

We assume that the coefficients of a differential operator $L(u)$ satisfy conditions of the type (7.60) (for an arbitrary non-negative u) and the condition $a(t, u)\geqslant 0$. Then the operator L^{-1} will leave the cone of non-negative functions invariant. In fact, if $f(t)\geqslant 0$ and $u^*(t)$ is a solution of the problem (7.98), then $u^*(t)$ will be, simultaneously, a solution of the problem

$$L(u^*)u=f(t), \quad u(t)|_{t\in\Gamma}=0\,, \tag{7.99}$$

and the solution of the last problem, as was explained in subsection 7.2.1, is non-negative.

The boundary value problem (7.96)–(7.62) can now be reduced to an equation with an operator which leaves the cone invariant by means of the substitution $L(u)u=w$.

Applications of the plan described require the overcoming of several difficulties. Here, for the time being, only preliminary results have been obtained.

7.3. Existence of positive periodic solutions for a system of ordinary differential equations of the first order

The general theory developed in the present book can be applied in two ways in the investigation of problems concerning periodic solutions of systems of ordinary differential equations.

We can first attempt to find in a phase space cones which are invariant with respect to the displacements along trajectories of the system. Moreover, if we can succeed in showing that for a displacement, corresponding to a change of time by the magnitude of the period of the right members, there is a fixed point, then from this one can deduce the presence of periodic solutions.

The second way is based on a transition from a system of differential equations to integral equations or integro-functional equations whose solutions correspond to periodic solutions of the system of differential equations. The system of integral equations or integro-functional equations is considered as an operator equation $\varphi=A\varphi$ in some Banach space E. If we succeed in finding in E a cone K which is invariant for A then we

can attempt, by means of general principles, to prove the presence of fixed points for the operator A and to investigate the dependence of fixed points on various parameters and so on. Conditions, for which there are fixed points for the operator A, are simultaneously the conditions for the existence of periodic solutions for the system of differential equations. The first of the indicated methods is considered in the present section; the second is considered in section 7.5.

7.3.1. *Poincaré's principle*

We consider the system of ordinary differential equations

$$\begin{aligned} &\frac{dx_1}{dt} = f_1(t, x_1, \ldots, x_n), \\ &\ldots\ldots\ldots\ldots \\ &\frac{dx_n}{dt} = f_n(t, x_1, \ldots, x_n). \end{aligned} \tag{7.100}$$

We shall assume that the right members are defined for all values of the variables:

$$-\infty < t, \quad x_1, \ldots, x_n < \infty .$$

We shall further assume that the right members are continuous with respect to the collection of variables and have the properties which guarantee the uniqueness of the solution of the Cauchy problem for the system (7.100).

In the sequel, we shall write the system (7.100) in the vector form

$$\frac{dx}{dt} = f(t, x). \tag{7.101}$$

The equation (7.101) is considered in n-dimensional space in which we define the norm by the equality

$$\|x\| = |x_1| + \ldots + |x_n| . \tag{7.102}$$

In this section, the question concerning the existence for system (7.100) of periodic solutions of a given period ω is considered. It is natural that the right members are also assumed to be periodic with respect to t:

$$f_i(t+\omega, x_1, \ldots, x_n) \equiv f_i(t, x_1, \ldots, x_n) \qquad (i=1, \ldots, n). \tag{7.103}$$

For the proof of the existence of periodic solutions in a number of cases, it is helpful in achieving success if we make use of the following geometrical principle of Poincaré.

We assume that every solution of the system (7.101) can be extended to the entire closed interval $0 \leqslant t \leqslant \omega$ (one must keep in mind that this assumption is an essential restriction). With respect to every point $x_0 \in E^n$ we construct the solution $x = x(t, x_0)$ of the system (7.101) which satisfies the initial condition

$$x(0, x_0) = x_0 . \tag{7.104}$$

We define the operator A by the equality

$$Ax_0 = x(\omega, x_0) . \tag{7.105}$$

If it is shown that the operator A has a fixed point x^*, then this means that for the system (7.101) there is a solution $x^*(t)$, satisfying the condition $x^*(\omega) = x^*(0) = x^*$. It follows from the periodicity with respect to t of the right members of the system that $x^*(t)$ can be extended with respect to periodicity to all values of t and this extension will be a solution of the system (7.101). Generally speaking, the existence of a periodic solution for the system (7.101) follows from the existence of a fixed point for the operator (7.105).

In a number of cases, it is convenient to consider the operator (7.105) not on the entire space E^n, but one some set $T \subset E^n$. The role of this set T will be played below by the cone K of vectors with non-negative coordinates. If the extendibility of all solutions of the system (7.101) cannot be guaranteed to the interval $[0, \omega]$, then it is sometimes convenient to construct a new system

$$\frac{dx}{dt} = g(t, x) , \tag{7.106}$$

whose solutions are extendible to $[0, \omega]$ where all periodic solutions of the new system are simultaneously solutions of the system (7.101). For the proof of the existence of periodic solutions for the system (7.106), the Poincaré principle already studied above can be applied.

7.3.2. *Existence of a non-negative periodic solution*

We further require that every function $f_i(t, x_1, \ldots, x_n)$ satisfy the condition

$$f_i(t, x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) \geqslant 0 \qquad (x_j \geqslant 0 \quad \text{for} \quad j \neq i) . \tag{7.107}$$

It follows from this condition that for movement along trajectories of system (7.101) not a single point can fall outside of the cone K.

In order not to refer to theorems concerning differential inequalities, we give a direct proof of this fact.
We consider the auxiliary system

$$\begin{aligned} \frac{dx_1}{dt} &= f_1(t, x_1, \ldots, x_n)+\varepsilon\,, \\ &\cdots\cdots\cdots\cdots \\ \frac{dx_n}{dt} &= f_n(t, x_1, \ldots, x_n)+\varepsilon\,, \end{aligned} \tag{7.108}$$

where ε is a fixed positive number. The trajectories of system (7.108) do not fall outside the cone K since at every point of the intersection of the trajectory with the boundary of the cone, the coordinates which are equal to zero are strictly increasing. It follows from the continuous dependence of the solutions of system (7.108) on the parameter ε and from the closure of the cone that the solutions of system (7.101) also do not fall outside the cone.
Let $\Phi(x_1, \ldots, x_n)$ be a continuously differentiable non-negative function satisfying the condition

$$\lim_{x_i\geqslant 0,\ \|x\|\to\infty} \Phi(x_1, \ldots, x_n)=\infty \tag{7.109}$$

and such that its level surfaces are convex. It is convenient to assume that the gradient of the function $\Phi(x_1, \ldots, x_n)$ does not become zero for sufficiently large values of $|x_1|+\ldots+|x_n|$.
We assume that

$$\sum_{i=1}^{n} f_i(t, x_1, \ldots, x_n)\frac{\partial}{\partial x_i}\Phi(x_1, \ldots, x_n)\leqslant 0 \tag{7.110}$$

if $x_1, \ldots, x_n\geqslant 0$ and

$$x_1+\ldots+x_n\geqslant R_0\,,$$

where R_0 is some positive number. Then every solution of system (7.101), satisfying the initial condition of the cone K, will be bounded and, consequently, will be defined for all positive numbers t. Let the norm of all the points x lying on the surface

$$\Phi(x_1, \ldots, x_n)=\Phi_0$$

be greater than R_0; such a Φ_0 can be chosen by virtue of (7.109). We denote by T the set of those points of the cone K on which the values of the function $\Phi(x_1, \ldots, x_n)$ do not exceed Φ_0.

No trajectory of the system (7.101) falls outside of the domain T. Therefore, the operator (7.105) transforms T into itself. The set T is bounded, convex and closed. It follows from Brouwer's fixed point principle (whose generalization is the above mentioned Schauder principle) that A has at least one fixed point on T.

Thus, the existence of at least one periodic solution of system (7.100), lying in the cone K, follows from conditions (7.107) and (7.110). As an example, we consider the case when

$$\Phi(x_1, \ldots, x_n) = \tfrac{1}{2}(x_1^2 + \ldots + x_n^2) . \tag{7.111}$$

Then the conditions for the existence of periodic solutions have the following form:

Theorem 7.16. *Let the right members of the system* (7.100) *be periodic with respect to* t *with period* ω. *Let conditions* (7.107) *be satisfied. Finally, let*

$$\sum_{i=1}^{n} x_i f_i(t, x_1, \ldots, x_n) \leqslant 0 \tag{7.112}$$

if the sum $x_1 + \ldots + x_n (x_j \geqslant 0)$ *is sufficiently large.*

Then the system (7.100) *has at least one non-negative periodic solution*

$$x_i(t) \geqslant 0 , \quad x_i(t+\omega) \equiv x_i(t) \qquad (i = 1, \ldots, n) . \tag{7.113}$$

The arguments of the present subsection carry over without difficulty to the case of a wider class of functions $\Phi(x_1, \ldots, x_n)$ satisfying condition (7.109).

The choice of distinct functions reduces to distinct conditions of the type (7.112). In particular, condition (7.112) can be replaced by the inequality

$$\sum_{i-1}^{n} x_i^{\alpha} f_i(t, x, \ldots, x_n) \leqslant 0 , \tag{7.114}$$

where α is an arbitrary fixed number of the interval $(-1, \infty)$.

7.3.3. *Formulation of the problem concerning the existence of a positive periodic solution*

In this and in following subsections, we shall assume that

$$f_i(t, 0, \ldots, 0) \equiv 0 \qquad (i = 1, \ldots, n) . \tag{7.115}$$

In this case, the system (7.100) has the obvious zero periodic solution. The question of the existence of a second periodic solution in the cone will be of interest to us.

Conditions of the type (7.112) or (7.114), as we have already indicated, guarantee the extendibility of all solutions of the system (7.100) with initial conditions on the cone K to all values $t \geqslant 0$. Therefore, the operator A is defined on the entire cone K and transforms K into itself. The assumption (7.115) means that $A\theta = \theta$. Theorems 4.11–4.16 can be applied for the proof of the existence of non-zero fixed points for the operator A.
It follows directly either from condition (7.112) or from analogous conditions that $Ax \gtrdot x$ for $x \in K$ and for sufficiently large values $\|x\|$, or it follows that operator A on elements with a large norm can be overdetermined in order for no new fixed points to appear for it and that the overdetermined operator $\tilde{A}$ will satisfy the condition $\tilde{A}x \gtrdot x$ on elements $x \in K$ with a sufficiently large norm. Therefore, the conditions for the existence of periodic solutions can be obtained in the following way: find the derivative $A'(\theta)$ of the operator A and require that the operator $A'(\theta)$ have a unique characteristic vector in the cone K to which there corresponds a characteristic value greater than 1. It turns out that the operator $A'(\theta)$ can be constructed as a solution of some linear system of differential equations with periodic coefficients.

7.3.4. *About linear systems of ordinary differential equations*
We consider the system

$$\begin{aligned} \frac{dx_1}{dt} &= b_{11}(t)x_1 + \ldots + b_{1n}(t)x_n + h_1(t), \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ \frac{dx_n}{dt} &= b_{n1}(t)x_1 + \ldots + b_{nn}(t)x_n + h_n(t), \end{aligned} \tag{7.116}$$

which we write in vector form as

$$\frac{dx}{dt} = B(t)x + h(t). \tag{7.117}$$

We shall assume that the functions $b_{ij}(t)$ and $h_i(t)$ are continuous.
In the construction and investigation of the solutions of the system (7.16), the so-called *matrixant* $U(t, s)$ defined by the equality

$$U(t, s) = I + \int_s^t B(t_1)\,dt_1 + \int_s^t B(t)\int_s^{t_2} B(t_1)\,dt_1\,dt_2 + \ldots, \tag{7.118}$$

which is a linear operator for fixed numbers t and s, plays an important

role since the series (7.118) converges uniformly with respect to the norm of the operators with respect to t and s which are from arbitrary finite intervals.

The solution of the linear homogeneous equation

$$\frac{dx}{dt} = B(t)x\,, \tag{7.119}$$

satisfying the initial condition

$$x(s)=x_0\,, \tag{7.120}$$

can be written in the form

$$x(t)=U(t, s)x_0\,. \tag{7.121}$$

The solution of the non-homogeneous equation (7.117) which satisfies the initial condition (7.120) is written in the form

$$x(t)=U(t, s)x_0+\int_s^t U(t, \tau)h(\tau)\,d\tau\,. \tag{7.122}$$

Both of the last formulas are verified by immediate differentiation if the equalities

$$U(t, t)=I\,,\quad U(t, \tau)U(\tau, s)=U(t, s)$$

are taken into consideration.

In the sequel, we shall assume that the coefficients $b_{ij}(t)$ of the system (7.116) are periodic with repect to t with period ω. For this particular case, the matrix

$$U=U(\omega, 0)$$

is called a *matrix of monodromy*

We introduce the notation

$$\max_{0\leqslant t\leqslant\omega} \|B(t)\|=b_0$$

and

$$\max_{0\leqslant t, s\leqslant\omega} \|U(t, s)\|=b_1\,.$$

Then the estimate

$$\|x(t)\| \leqslant b_1\|x_0\|+b_1\omega \max_{0\leqslant\tau\leqslant\omega} \|h(\tau)\| \qquad (0\leqslant t\leqslant\omega) \tag{7.123}$$

follows from (7.122).

Matrixants will be considered below of linear systems whose coefficients

satisfy the additional condition:

$$b_{ij}(t) \geqslant 0 \qquad (0 \leqslant t \leqslant \omega,\ i, j = 1, \ldots, n,\ i \neq j)\,. \tag{7.124}$$

As we already know, the solutions of the homogeneous system (7.119) under the condition (7.124) do not fall outside of the cone K. This means that the matrixant is an operator which leaves a cone invariant. We remark that it is rather difficult to deduce the invariance of a cone directly from the form (7.118) of the matrixant $U(t, s)$ since the diagonal elements $b_{ii}(t)$ are not assumed to be positive.
The next theorem follows from a result of chapter 2.

Theorem 7.17. *If conditions* (7.124) *are satisfied, then the matrix of monodromy has at least one non-negative characteristic value. A characteristic vector with non-negative coordinates corresponds to this characteristic value* λ_0.
If the functions $b_{ij}(t)$ *for* $i \neq j$ *are positive, then* λ_0 *is a simpled characteristic value of the matrix of monodromy; the remaining characteristic values are in absolute value less than* λ_0.

In conclusion, we consider the case of system (7.119) with constant coefficients, that is, the system

$$\begin{aligned} \frac{dx_1}{dt} &= b_{11}x_1 + \ldots + b_{1n}x_n\,, \\ &\ldots\ldots\ldots\ldots\ldots \\ \frac{dx_n}{dt} &= b_{n1}x_1 + \ldots + b_{nn}x_n\,. \end{aligned} \tag{7.125}$$

The matrix $e^{\boldsymbol{B}(t-s)}$ is the matrixant of this system; the matrix of monodromy is the matrix $e^{\boldsymbol{B}\omega}$. The characteristic values of the matrix of monodromy are equal to $e^{\lambda\omega}$, where the λ's are the characteristic values of the matrix $\boldsymbol{B}$, that is, roots of the characteristic equation

$$\begin{vmatrix} b_{11}-\lambda & b_{12} & \ldots & b_{1n} \\ b_{21} & b_{22}-\lambda & \ldots & b_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ b_{n1} & b_{n2} & \ldots & b_{nn}-\lambda \end{vmatrix} = 0\,. \tag{7.126}$$

It follows from Theorem 7.17 that equation (7.126) has at least one real root if all the non-diagonal elements of the matrix $\boldsymbol{B}$ are non-negative (and this fact can, moreover, be easily obtained directly).

7.3.5. *Derivative of an operator A*

We assume that the functions $f_i(t, x_1, \ldots, x_n)$ satisfy the conditions

$$f_i(t, 0, \ldots, 0) \equiv 0 \qquad (i=1, \ldots, n)\,. \tag{7.127}$$

We shall assume that the derivatives

$$b_{ij}(t) = \frac{\partial}{\partial x_j} f_i(t, 0, \ldots, 0) \qquad (i, j=1, \ldots, n)$$

exist and, moreover, that the representations

$$f_i(t, x_1, \ldots, x_n) = b_{i1}(t)x_1 + \ldots + b_{in}(t)x_n + \varphi_i(t, x_1, \ldots, x_n) \tag{7.128}$$

hold, where the functions $\varphi_i(t, x_1, \ldots, x_n)$ contain infinitesimals of higher orders:

$$|\varphi_i(t, x_1, \ldots, x_n)| \leqslant \alpha(\|x\|) \qquad (i=1, 2, \ldots, n) \tag{7.129}$$

and the non-decreasing function $\alpha(\varrho)$ satisfies the condition

$$\lim_{\varrho \to 0} \frac{\alpha(\varrho)}{\varrho} = 0\,. \tag{7.130}$$

Then the system (7.101) can be written in the form

$$\frac{dx}{dt} = \boldsymbol{B}(t)x + \varphi(t, x)\,, \tag{7.131}$$

where

$$\|\varphi(t, x)\| \leqslant n\alpha(\|x\|)\,. \tag{7.132}$$

We shall show that the derivative $A'(\theta)$ of the operator (7.105) is defined by the equality

$$A'(\theta)x_0 = \boldsymbol{U}x_0 = \boldsymbol{U}(\omega, 0)x_0\,. \tag{7.133}$$

The equation (7.131) is equivalent to the integral equation

$$x(t) = \boldsymbol{U}(t, 0)x_0 + \int_0^t \boldsymbol{U}(t, \tau)\varphi[\tau, x(\tau)]\,d\tau\,,$$

from which the estimate

$$\|x(t)\| \leqslant b_1\|x_0\| + b_1\omega \max_{0 \leqslant \tau \leqslant \omega} \|\varphi[\tau, x(\tau)]\| \qquad (0 \leqslant t \leqslant \omega)$$

follows and, by virtue of (7.132),

$$\max_{0 \leqslant t \leqslant \omega} \|x(t)\| \leqslant b_1\|x_0\| + b_1\omega n\alpha\left(\max_{0 \leqslant \tau \leqslant \omega} \|x(\tau)\|\right)\,. \tag{7.134}$$

It follows from (7.130) and the continuous dependence of the solution on the initial conditions that there exists a δ_0 such that for $\|x_0\| < \delta_0$ the inequality

$$\alpha\left(\max_{0\leqslant\tau\leqslant\omega}\|x(\tau)\|\right) \leqslant \frac{1}{2b_1\omega n}\max_{0\leqslant\tau\leqslant\omega}\|x(\tau)\|$$

is satisfied. This means that for $\|x_0\| < \delta_0$ the following estimate

$$\max_{0\leqslant t\leqslant\omega}\|x(t)\| \leqslant 2b_1\|x_0\| \tag{7.135}$$

follows from (7.134).

We denote by $y(t)$ a solution of the homogeneous system $dy/dt = \boldsymbol{B}(t)y$. We shall estimate the difference $x(t)-y(t)$, where the solution $x(t)$ satisfies the same initial condition as is satisfied by $y(t)$. Obviously

$$\frac{d[x(t)-y(t)]}{dt} = \boldsymbol{B}(t)[x(t)-y(t)]+\varphi[t, x(t)]\,,$$

from which it follows that

$$x(t)-y(t) = \int_0^t \boldsymbol{U}(t, \tau)\varphi[\tau, x(\tau)]\,d\tau$$

and, by virtue of (7.135), the inequality

$$\|x(t)-y(t)\| \leqslant b_1\omega n\alpha(2b_1\|x_0\|) \qquad (0\leqslant t\leqslant\omega)$$

holds for $\|x_0\| < \delta_0$. It then follows from (7.130) that

$$\lim_{\|x_0\|\to 0}\max_{0\leqslant t\leqslant\omega}\frac{\|x(t)-y(t)\|}{\|x_0\|} \leqslant b_1\omega n\lim_{\|x_0\|\to 0}\frac{\alpha(2b_1\|x_0\|)}{\|x_0\|} = 0\,.$$

This completes the proof of the formula (7.133).

7.3.6. *Existence of a positive periodic solution*

In subsection 7.3.3, it was shown that under conditions of the type (7.12), the existence of a positive periodic solution can be guaranteed if the operator $\boldsymbol{A}'(\theta)$ has a unique characteristic vector in the cone to which there corresponds a characteristic value greater than 1.

Thus, we have proved the following theorem.

Theorem 7.18. *Suppose the functions $f_i(t, x_1, \ldots, x_n)$, which satisfy the conditions* (7.107) *and* (7.117), *allow the representation* (7.128). *Let condition* (7.112) *be satisfied. Finally, let there exist a greatest real characteristic value of the matrix of monodromy of the linear system $dy/dt = \boldsymbol{B}(t)y$*

which is greater than 1 *and let the matrix in the cone of non-negative vectors not have non-zero fixed points* (*and not have positive characteristic vectors to which a characteristic value equal to* 1 *corresponds*).
Then the system (7.100) *has at least one positive periodic solution* $x(t)$:

$$x_i(t) \geqslant 0\,, \quad \sum_{i=1}^{n} x_i(t) > 0\,, \quad x_i(t+\omega) \equiv x_i(t)\,.$$

We remark that the non-negativeness of the functions $b_{ij}(t)$ for $i \neq j$ follows from conditions (7.107) and (7.117). Therefore, the matrix of monodromy always has a positive real characteristic value.
If the matrix $\boldsymbol{B}(t)$ does not depend on t, then a sufficient condition for the existence of a positive periodic solution (provided the conditions (7.107), (7.117) and (7.112) are fulfilled) is formulated in a quite easy fashion—it suffices that the equation (7.126) have a positive root and no negative root.
We now assume that for non-negative x_j and for sufficiently large sums $x_1 + \ldots + x_n$ the inequality

$$\sum_{i=1}^{n} f_i(t, x_1, \ldots, x_n) \frac{\partial}{\partial x_i} \Phi(x_1, \ldots, x_n) \geqslant 0 \tag{7.136}$$

is satisfied in a definite sense to that of inequality (7.110).
If condition (7.136) is satisfied, then the solution of system (7.100) might not be extendible to the interval $[0, \omega]$. For this case, along with the system (7.100), we consider the auxiliary system

$$\begin{aligned} \frac{dx_1}{dt} &= \frac{g_1(t, x_1, \ldots, x_n)}{1 + \varphi(x_1, \ldots, x_n) \sum_{i=1}^{n} |g_i(t, x_1, \ldots, x_n)|}\,, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{dx_n}{dt} &= \frac{g_n(t, x_1, \ldots, x_n)}{1 + \varphi(x_1, \ldots, x_n) \sum_{i=1}^{n} |g_i(t, x_1, \ldots, x_n)|}\,, \end{aligned} \tag{7.137}$$

where

$$g_i(t, x_1, \ldots, x_n) = \begin{cases} f_i(t, x_1, \ldots, x_n), & \text{if} \quad \Phi(x_1, \ldots, x_n) \leqslant \Phi_0\,, \\ f_i(t, x_1, \ldots, x_n) + \varphi(x_1, \ldots, x_n) \dfrac{\partial}{\partial x_i} \Phi(x_1, \ldots, x_n)\,, & \\ & \text{if} \quad \Phi(x_1, \ldots, x_n) \geqslant \Phi_0\,, \end{cases}$$

and

$$\varphi(x_1, \ldots, x_n) = \begin{cases} 0, & \text{if } \Phi(x_1, \ldots, x_n) \leqslant \Phi_0, \\ \Phi(x_1, \ldots, x_n) - \Phi_0, & \text{if } \Phi(x_1, \ldots, x_n) \geqslant \Phi_0, \end{cases}$$

and Φ_0 is a positive number such that for

$$\Phi(x_1, \ldots, x_n) \geqslant \Phi_0 \tag{7.138}$$

the inequality (7.136) is satisfied.

The right members of system (7.137) are bounded (in the cone of non-negative vectors); therefore, its solutions can be assumed to be defined for all $t \geqslant 0$. Every non-negative periodic solution

$$x^*(t) = \{x_1^*(t), \ldots, x_n^*(t)\}$$

of the system (7.137) will be simultaneously a periodic solution of the basic system (7.100). In order to convince ourselves of this, it suffices to remark that by virtue of (7.136) on every solution

$$x(t) = \{x_1(t), \ldots, x_n(t)\}$$

of system (7.137), lying in the domain (7.138), the function

$$u(t) = \Phi[x_1(t), \ldots, x_n(t)]$$

is strictly increasing since its derivative is positive.

The right members of system (7.137) satisfy the condition (7.136). Thus, the generality of the result is not diminished if we further assume for condition (7.136) that all solutions of the system (7.100) are defined on the interval $[0, \omega]$.

It is easy to deduce from condition (7.136) the corollary that the operator A, defined by the equality (7.105), either satisfies the condition

$$Ax_0 \leqq x_0,$$

for large values of $B(x_1, \ldots x_n)$ or it can be overdetermined in order that this condition be satisfied for the overdetermined operator. It then follows from the results of chapter 4 that the assumption that the spectrum of the operator $A'(\theta)$ lies in a circle whose radius is less than 1 is a sufficient condition for the existence of periodic solutions for system (7.100). As we already know, the operator $A'(\theta)$ is the operator of the monodromy of a linearized system.

Choosing as $\Phi(x_1, \ldots, x_n)$ the function (7.111), we arrive at the following theorem:

Theorem 7.19 *Let the function* $f_i(t, x_1, \ldots, x_n)$ *which satisfies conditions* (7.107) *and* (7.117) *allow the representation* (7.128). *Let the condition*

$$\sum_{i=1}^{n} x_i f_i(t, x_1, \ldots, x_n) \geqslant 0 \tag{7.139}$$

be satisfied if the sum $x_1 + \ldots + x_n (x_j \geqslant 0)$ *is sufficiently large. Finally, let all the characteristic values of the matrix of monodromy of the linearized system* $dy/dt = \boldsymbol{B}(t)y$ *be in absolute value less than* 1.
Then the system (7.100) *has at least one positive periodic solution* $x(t)$:

$$x_i(t) \geqslant 0\,, \quad \sum_{i=1}^{n} x_i(t) > 0\,, \quad x_i(t+\omega) \equiv x_i(t)\,.$$

7.3.7. *Linearization at infinity*
We assume that the functions $f_i(t, x_1, \ldots, x_n)$ are asymptotically linear with respect to the cone K. This means that functions $c_{ij}(t)$ exist such that for non-negative $x_1, \ldots, x_n$ the representations

$$f_i(t, x_1, \ldots, x_n) = c_{i1}(t)x_1 + \ldots + c_{in}(t)x_n + \psi_i(t, x_1, \ldots, x_n) \tag{7.140}$$

are valid where

$$|\psi_i(t, x_1, \ldots, x_n)| < \gamma(\|x\|)\,, \tag{7.141}$$

where the non-decreasing function $\gamma(\varrho)$ satisfies the condition

$$\lim_{\varrho \to \infty} \frac{\gamma(\varrho)}{\varrho} = 0\,. \tag{7.142}$$

In the sequel, the functions $c_{ij}(t)$ are assumed to be continuous.
If condition (7.140) is satisfied, then the system (7.101) can be rewritten in the form

$$\frac{dx}{dt} = \boldsymbol{C}(t)x + \psi(t, x)\,, \tag{7.143}$$

where

$$\|\psi(t, x)\| \leqslant n\gamma(\|x\|)\,. \tag{7.144}$$

We note that all the solutions of the system (7.143), lying in the cone, will be defined for all values $t \geqslant 0$.
The matrixant of the linear system

$$\frac{dy}{dt} = \boldsymbol{C}(t)y \tag{7.145}$$

is denoted by $V(t, s)$. Then the solution of the system (7.143) will satisfy the integral equation

$$x(t) = V(t, 0)x_0 + \int_0^t V(t, \tau)\psi[\tau, x(\tau)]d\tau . \tag{7.146}$$

Our immediate aim is the proof of the asymptotic linearity with respect to a cone of the operator A, defined by the equality (7.105), and the establishment of the equalities

$$A'(\infty)x_0 = Vx_0 \equiv V(\omega, 0)x_0 . \tag{7.147}$$

Let an arbitrary number $\varepsilon > 0$ be given. The existence of a number $M = M(\varepsilon)$ such that

$$\|\psi(t, x)\| \leqslant \varepsilon\|x\| + M(\varepsilon) \qquad (x \in K) \tag{7.148}$$

follows from (7.144) and (7.142). Then (7.146) implies the inequality

$$\max_{0\leqslant t\leqslant\omega} \|x(t)\| \leqslant b_2\|x_0\| + b_2\omega[\varepsilon \max_{0\leqslant t\leqslant\omega} \|x(t)\| + M(\varepsilon)] ,$$

from which it follows that

$$\max_{0\leqslant t\leqslant\omega} \|x(t)\| \leqslant \frac{b_2\|x_0\| + b_2\omega M(\varepsilon)}{1 - b_2\omega\varepsilon} , \tag{7.149}$$

where

$$b_2 = \max_{0\leqslant t, s\leqslant\omega} \|V(t, s)\| .$$

Let $y(t)$ be a solution of system (7.145), satisfying the initial condition which is satisfied by the solution $x(t)$ of system (7.143). It is obvious that

$$\frac{d(x-y)}{dt} = C(t)(x-y) + \psi(t, x) ,$$

from which it follows that

$$x(t) - y(t) = \int_0^t V(t, \tau)\psi[\tau, x(\tau)]d\tau$$

and, by virtue of (7.148) and (7.149),

$$\|x(t) - y(t)\| \leqslant b_2\omega\varepsilon \frac{b_2\|x_0\| + b_2\omega M(\varepsilon)}{1 - b_2\omega\varepsilon} .$$

It follows from the inequalities which were obtained that

$$\lim_{x_0\in K,\ \|x_0\|\to\infty} \frac{\|x(t) - y(t)\|}{\|x_0\|} \leqslant \frac{b_2^2\omega\varepsilon}{1 - b_2\omega\varepsilon} ,$$

and since ε is arbitrary, then

$$\lim_{x_0 \in K,\ \|x_0\| \to \infty} \frac{\|x(t)-y(t)\|}{\|x_0\|} = 0 .$$

Setting $t=\omega$ in the last equality, we arrive at equality (7.147).

7.3.8. *Existence of positive periodic solutions for systems which are linear at infinity*

It follows from the results of the last subsection and from Theorem 4.7 that system (7.100) has at least one periodic solution if the condition

$$f_i(t, x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) \geqslant 0 \qquad (x_j \geqslant 0 \text{ for } j \neq i) \tag{7.107}$$

is satisfied and if all characteristic values of the matrix of monodromy of system (7.145) are in absolute value less than 1. It is assumed here that the right members of the system allow the representation (7.140).

We formulate as a separate theorem the corollaries to the Theorems 4.11 and 4.16.

Theorem 7.20. *Let the conditions* (7.107) *and* (7.117) *be satisfied. Let the right members of system* (7.100) *allow the representation* (7.128) *and* (7.140) *where every matrix of monodromy of the systems* $dy/dt = B(t)y$ *and* $dy/dt = C(t)y$ *has a unique characteristic vector with non-negative components to which there correspond characteristic values* λ_0 *and* λ_∞.

Then it suffices for the existence of a positive solution that one of the following conditions be satisfied: either

$$\lambda_0 < 1 , \quad \lambda_\infty > 1 ,$$

or

$$\lambda_0 > 1 , \quad \lambda_\infty < 1 .$$

7.3.9. *Remark about uniqueness theorems*

Uniqueness theorems and theorems about the convergence of successive approximations established in chapter 6 are also applicable to the problem about periodic solutions in the system (7.100). These applications are based on criteria for concavity of the operators of pointwise transformations.

We note that the conditions for applicability of the method of successive approximations are simultaneously conditions for asymptotic stability of a corresponding periodic solution.

Corresponding results will be studied in another place.

7.4. Two point boundary value problem

7.4.1. *Green's function for the operator* $\ddot{x}$

We consider the boundary value problem

$$-\frac{d^2x}{dt^2} = y(t), \quad x(0)=x(1)=0\,. \tag{7.150}$$

The function $x(t)$ is defined, as it is known, by the equality

$$x(t)=Ay(t)=\int_0^1 G(t,s)y(s)\,, \tag{7.151}$$

where Green's function $G(t, s)$ has the form

$$G(t,s) = \begin{cases} t(1-s)\,, & \text{if} \quad t\leqslant s\,, \\ s(1-t)\,, & \text{if} \quad s\leqslant t\,. \end{cases} \tag{7.152}$$

The validity of formula (7.151) is easily verified by direct differentiation.

Lemma 7.6. *Let* $y(t)\geqslant 0$ *and* $y(t)\not\equiv 0$. *Then positive numbers* α *and* β *can be found such that*

$$\alpha t(1-t)\leqslant Ay(t)\leqslant \beta t(1-t) \qquad (0\leqslant t\leqslant 1)\,. \tag{7.153}$$

Proof. Let $[t_1, t_2]$ be an arbitrary segment which is a part of the segment [0, 1]. By virtue of (7.152), the inequality

$$\int_{t_1}^{t_2} G(t,s)\,ds \geqslant \int_{(t_1+t_2)/2}^{t_2} G(t,s)\,ds = t\int_{(t_1+t_2)/2}^{t_2}(1-s)\,ds \geqslant$$

$$\geqslant t\int_{1-(t_2-t_1)/2}^{1}(1-s)\,ds = at$$

will be satisfied for $0\leqslant t\leqslant (t_1+t_2)/2$ and the inequality

$$\int_{t_1}^{t_1} G(t,s)\,ds \geqslant \int_{t_1}^{(t_1+t_2)/2} G(t,s)\,ds =$$

$$=(1-t)\int_{t_1}^{(t_1+t_2)/2} s\,ds \geqslant (1-t)\int_0^{(t_2-t_1)/2} s\,ds = a(1-t)$$

will be satisfied for $(t_1+t_2)/2\leqslant t\leqslant 1$.

It follows from the inequalities obtained that the inequality

$$\int_{t_1}^{t_2} G(t,s)\,ds \geqslant a\,\min\{t, 1-t\} \geqslant at(1-t)$$

is valid for all $t \in [0, 1]$.
Obviously,

$$\int_0^1 G(t, s)\,ds = \int_0^t s(1-t)\,ds + \int_t^1 t(1-s)\,ds = \tfrac{1}{2}t(1-t)\,.$$

Thus,

$$\int_{t_1}^{t_2} G(t, s)\,ds \geqslant \eta \int_0^1 G(t, s)\,ds\,.$$

The last inequality is the condition (7.4) of Lemma 7.1. The validity of the inequalities (7.153) follows from this lemma.
This completes the proof of the lemma.

The operator (7.151) acts in the space C of continuous functions on [0, 1] and its values are differentiable functions. We set

$$\boldsymbol{B}y(t) = \frac{d}{dt}\boldsymbol{A}y(t)\,.$$

Obviously,

$$\boldsymbol{B}y(t) = \int_0^1 G_t'(t, s)y(s)\,ds = -\int_0^t sy(s)\,ds + \int_t^1 (1-s)y(s)\,ds\,. \tag{7.154}$$

The operators $\boldsymbol{A}$ and $\boldsymbol{B}$ are completely continuous in the space C. It follows from the non-negativeness of $G(t, s)$ that the operator $\boldsymbol{A}$ leaves the cone K of non-negative functions invariant. In particular, Lemma 7.6 means that operator (7.151) is u_0-positive, where

$$u_0(t) = 2\int_0^1 G(t, s)\,ds = t(1-t)\,. \tag{7.155}$$

7.4.2. *Characteristic values and the oscillation of solutions*
In the following constructions, an important role will be played by operaters $\boldsymbol{P}$ and $\boldsymbol{Q}$ of the form

$$\boldsymbol{P}x(t) = b(t)\int_0^1 G(t, s)x(s)\,ds \tag{7.156}$$

and

$$\boldsymbol{Q}x(t) = \int_0^1 G(t, s)b(s)x(s)\,ds\,. \tag{7.157}$$

The function $b(t)$ is assumed to be non-negative.
Each of the operators $\boldsymbol{P}$ and $\boldsymbol{Q}$ evidently is v_0-positive. This follows from

the inequalities (7.153) if, for the case of the operator $\boldsymbol{P}$, we set $v_0(t)=b(t)u_0(t)$ and, for the case of the operator $\boldsymbol{Q}$, we set $v_0(t)=u_0(t)$. Therefore, each of the operators (7.156) and (7.157) has a unique non-negative characteristic function to which there corresponds a simple positive characteristic value where this characteristic value is greater than the absolute magnitudes of the remaining characteristic values. This characteristic value will be denoted by $\lambda(\boldsymbol{P})$ or $\lambda(\boldsymbol{Q})$ respectively.

The operators $\boldsymbol{P}$ and $\boldsymbol{Q}$ are closely connected with the boundary value problem

$$\lambda \frac{d^2x}{dt^2} + b(t)x=0\,, \quad x(0)=x(1)=0\,. \tag{7.158}$$

This boundary value problem is equivalent to the equation

$$\boldsymbol{Q}x(t)=\lambda x(t)\,.$$

If the substitution (7.150) is made in the boundary value problem, then we arrive at the equation

$$\boldsymbol{P}x(t)=\lambda x(t)\,.$$

Thus, the characteristic values of the operators $\boldsymbol{P}$ and $\boldsymbol{Q}$ coincide with characteristic values of the boundary value problem (7.158).

Let λ^* be the greatest characteristic value of the boundary value problem (7.158). The number λ^* will be, simultaneously, the greatest characteristic value of the operator $\boldsymbol{Q}$. Therefore, the corresponding characteristic function $x^*(t)$ is positive on the interval (0, 1). It follows from Sturm's theorem (see V. V. Stepanov [1]) that no solution of the equation

$$\lambda^* \frac{d^2x}{dt^2} + b(t)x=0$$

vanishes twice on the interval (0, 1).

We assume that $\lambda^*<1$. Then in the equation

$$\frac{d^2x}{dt^2} + \frac{b(t)}{\lambda^*}x=0 \tag{7.159}$$

the coefficient of x is greater than the coefficient $b(t)$ of x in the equation

$$\frac{d^2x}{dt^2} + b(t)x=0\,. \tag{7.160}$$

It follows from the comparison theorems (see V. V. Stepanov [1]) that there are no solutions for equation (7.160) which vanish twice on (0, 1).

Moreover, there are no solutions for equation (7.160) which vanish twice on [0, 1]. In fact, if there were a solution $x_1(t)$ for equation (7.160) which would become zero at the points 0 and t_1, where $0<t_1<1$, then the solution $x_2(t)$ which vanishes at the point $t=\varepsilon$ (where ε is a sufficiently

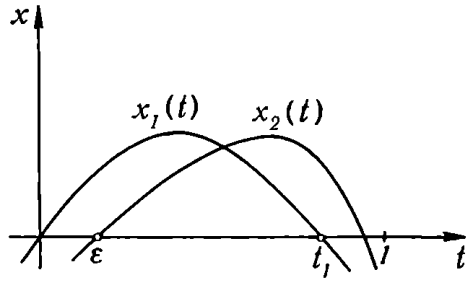


Fig. 5

small positive number) would become zero at some point $t_2 \in (t_1, 1)$ (Fig. 5). This would mean that there are solutions for equation (7.160), which vanish twice on (0, 1) which cannot be the case. Analogously, it is proved that there are no solutions of the equation (7.160) which vanish at the point $t=1$ and at some point $t_1 \in (0, 1)$. Finally, there cannot be solutions which vanish at the points 0 and 1—otherwise this solution would be of constant sign and it could be assumed non-negative without loss of generality; but, in this case, for the boundary value problem (7.158) or, what is the same, for the operator (7.157) there are two characteristic values $\lambda=\lambda^*$ and $\lambda=1$, to which there correspond non-negative characteristic functions, and this contradicts the fact that the operator (7.157) is v_0-positive.

We assume that $\lambda^*>1$. Then if follows from the comparison theorem that between every two zeros of an arbitrary solution of equation (7.159) an aribtary solution of equation (7.160) vanishes at least once. Let $x_1(t)$ be a solution of the equation (7.160), which vanishes at the point $t=0$; the following zero t_1 will be in the interval (0, 1) (see Fig. 5). The solution of equation (7.160) which vanishes at the point $t=\varepsilon$ (ε is a sufficiently small positive number) will vanish twice on the interval (0, 1).

Let $\lambda^*=1$. Then no solution of equation (7.160) (which in the case considered coincides with equation (7.159)) vanishes twice on the interval (0, 1), since otherwise the non-negative characteristic function by virtue

of Sturm's theorem would vanish at some point of the interval (0, 1) and would change sign.
We have thus proved the following theorem.

Theorem 7.21. *In order for a greatest characteristic value* λ^* *of the operators* $\boldsymbol{P}$ *and* $\boldsymbol{Q}$ *to satisfy the inequality* $\lambda^*<1$, *it is necessary and sufficient that no non-trivial solution of equation* (7.160) *vanish on the interval* [0, 1]. *In order for the greatest characteristic value* λ^* *of the operators* $\boldsymbol{P}$ *and* $\boldsymbol{Q}$ *to satisfy inequality* $\lambda^*>1$, *it is necessary and sufficient that at least one non-trivial solution of the equation* (7.160) *vanish twice on the interval* (0, 1).

7.4.3. *Two point boundary value problem for a scalar equation*
We consider the equation

$$\frac{d^2x}{dt^2}+f\left(t, x, \frac{dx}{dt}\right)=0\,. \tag{7.161}$$

The problem about the determination of solutions of the equation (7.161) satisfying the boundary conditions

$$x(a)=x_1\,,\quad x(b)=x_2$$

is called a two point boundary value problem. In other words, the solutions of a two point boundary value problem are integral curves passing through two given points.
After the change in variables

$$t=a+(b-a)\tau\,,\quad x=y+x_1+\frac{x_2-x_1}{b-a}(t-a)$$

the two point problem transforms into a problem concerning the determination of solutions of a differential equation of the second order, assuming zero values at the points 0 and 1. We assume at once that the solutions of equation (7.161) are sought which satisfy the boundary conditions

$$x(0)=x(1)=0\,. \tag{7.162}$$

The non-negative solutions of problems (7.161)–(7.162) will be of interest to us.
Making in equation (7.161) the substitution (7.150), we obtain an integral equation

$$y(t)=f\left[t, \int_0^1 G(t, s)y(s)\,ds\,, \quad \int_0^1 G_t'(t, s)y(s)\,ds\right], \tag{7.163}$$

which is equivalent to the boundary value problem (7.161)–(7.162) and which we write in the form

$$y(t)=f[t, Ay(t), By(t)]\,, \tag{7.163}$$

where A and B are the operators (7.151) and (7.154).
If $f(t, u, v)$ is continuous with respect to all its arguments, then the operator F defined by the right member of the equation (7.163),

$$Fy(t)=f[t, Ay(t), By(t)]\,, \tag{7.164}$$

acts in C and is completely continuous. In the sequel, it is assumed that

$$f(t, u, v)\geqslant 0 \qquad (0\leqslant t\leqslant 1\,,\ u\geqslant 0\,,\ -\infty<v<\infty)\,. \tag{7.165}$$

Then the operator (7.164) leaves the cone K of non-negative functions invariant.

7.4.4. *Non-negative solutions of a two point boundary value problem*
Let

$$0\leqslant f(t, u, v)\leqslant a+b_0(t)u \qquad (0\leqslant t\leqslant 1\,,\ u\geqslant 0\,,\ -\infty<v<\infty)\,. \tag{7.166}$$

Then the operator (7.164) will have a majorant

$$F^+y(t)=a+b_0(t)Ay(t)\,,$$

where F^+ will have a strong asymptotic derivative

$$F^{+\prime}(\infty)y(t)=b_0(t)Ay(t)\,.$$

It follows from Theorem 4.10 that equation (7.163) has at least one non-negative solution if the greatest characteristic value λ^* of the operator $F^{+\prime}(\infty)$ is less than 1. Therefore, the two point boundary value problem has at least one non-negative solution if $\lambda^*<1$. If Theorem 7.21 is used, then the last statement can be formulated in the following manner:

Theorem 7.22. *Let the continuous function $f(t, u, v)$ satisfy condition* (7.166). *Suppose no non-trivial solution of the equation*

$$\frac{d^2x}{dt^2}+b_0(t)x=0$$

vanishes twice on the interval [0, 1].
Then the boundary value problem (7.161)–(7.162) *has at least one non-negative solution.*

We cite one theorem about the existence of non-trivial non-negative solutions of the two point boundary value problem for the conditions when this problem has the trivial zero solution, that is, the equality

$$f(t, 0, 0)\equiv 0 \qquad (0\leqslant t\leqslant 1) \tag{7.167}$$

holds.

Let $f(t, u, v)$ be continuously differentiable with respect to u and v for small values of these variables. It follows from condition (7.165) that $f_v'(t, 0, 0)\equiv 0$. The operator (7.164) has a strong Fréchet derivative

$$F'(\theta)y(t)=a(t)Ay(t) \tag{7.168}$$

at the point θ, where $a(t)=f_u'(t, 0, 0)$ and $a(t)$ is a non-negative continuous function. Moreover, let the function $f(t, u, v)$ be asymptotically linear in the sense that

$$\lim_{u\to\infty} \sup_{0\leqslant t\leqslant 1,\ -\infty<v<\infty} \frac{|f(t, u, v)-b(t)u|}{u} = 0\,,$$

where $b(t)$ is some non-negative continuous function. It is not difficult to see (page 265, section 7.2, subsection 7.2.6) that under this condition the operator F has a strong asymptotic derivative

$$F'(\infty)y(t)=b(t)Ay(t)\,. \tag{7.169}$$

The results of chapter 4 together with Theorem 7.21 imply the following theorem.

Theorem 7.23. *Let condition* (7.167) *be satisfied. Suppose no non-trivial solution of one of the equations*

$$\frac{d^2x}{dt^2} + a(t)x=0\,, \quad \frac{d^2x}{dt^2} + b(t)x=0$$

vanishes twice on the interval [0, 1] *and that the other equation has at least one solution which vanishes on the interval* (0, 1).

Then the boundary value problem (7.161)–(7.162) *has, besides the zero solution, at least one more non-negative solution.*

We now consider a simpler two point problem:

$$\frac{d^2x}{dt^2} + f(x)=0\,, \quad x(0)=x(1)=0\,. \tag{7.170}$$

It is equivalent to the integral equation

$$x(t)=\int_0^1 G(t, s)f[x(s)]ds\,. \tag{7.171}$$

If the function $f(u)$ is concave, then the operator defined by the right member of the equation (7.171) will be u_0-concave, where $u_0(t)=t(1-t)$. Therefore, *the concavity of the function $f(u)$ implies the uniqueness of the non-trivial non-negative solution of problem* (7.170).

The reader can formulate without difficulty corollaries for the two point boundary value problem to the other results of chapters 4–6.

We note that the theorems mentioned above about the solutions of a two point boundary value problem for a scalar equation (7.161) can also be obtained without resorting to the methods of functional analysis.

To this end, the method of study of integral curves on a phase plane can be used. However, the applied methods have the serious advantage that they can carry over without difficulty to the case of vector equations. The reader can find examples of such a transfer in the following section.

We make one further remark. An analysis of the phase portrait of equation (7.170) shows that it has a unique non-trivial non-negative solution for the case when $f(u)$ ($f(0)=0$) is convex. In other words, the integral equation (7.171) with a convex function $f(u)$ has a unique non-trivial solution in the cone of non-negative functions. With what general properties of the convex operator, defined by the right member of the equation, is uniqueness caused in this case? This question has not yet been investigated.

7.4.5. *One further condition for the solvability of a two point boundary value problem*

We assume that

$$0\leqslant f(t, x, p)\leqslant\varphi(t)R(p) \qquad (0\leqslant t\leqslant 1,\ x\geqslant 0,\ -\infty<p<\infty),$$

and consider the operator (7.164). If this operator F does not have a fixed point on the cone K of non-negative functions, then a sequence $y_n(t)\in K$ can be found such that $\|y_n\|\to\infty$ and

$$Fy_n(t)\geqslant y_n(t) \qquad (n=1, 2, \ldots).$$

The last inequalities, after the substitution of (7.151), take the form

$$-\frac{d^2x_n(t)}{dt^2}\leqslant\varphi(t)R\left[\frac{dx_n(t)}{dt}\right] \qquad (n=1, 2, \ldots).$$

Let $-\ddot{x}_n(t_n)\to\infty$. It follows from the last inequalities that $\dot{x}_n(t_n)\to\infty$. Since $x_n(0)=x_n(1)=0$, then points t_n^* can be found such that $\dot{x}_n(t_n^*)=0$ ($n=1, 2, \ldots$). Obviously,

$$\left|\int_{t_n^*}^{t_n}\varphi(t)\,dt\right| \geqslant \left|\int_{t_n^*}^{t_n}\frac{\ddot{x}_n(t)\,dt}{R[\dot{x}_n(t)]}\right| = \left|\int_0^{x\,(t_n)}\frac{dp}{R(p)}\right|,$$

from which it follows that

$$\int_0^1\varphi(t)\,dt > \left|\int_0^{x_n(t_n)}\frac{dp}{R(p)}\right| \qquad (n=1, 2, \ldots)$$

and

$$\int_0^1\varphi(t)\,dt \geqslant \min\left\{\int_{-\infty}^0\frac{dp}{R(p)},\ \int_0^\infty\frac{dp}{R(p)}\right\}.$$

Thus, the *inequality*

$$\int_0^1\varphi(t)\,dt < \min\left\{\int_{-\infty}^0\frac{dp}{R(p)},\ \int_0^\infty\frac{dp}{R(p)}\right\}$$

is a sufficient condition for the solution of problem (7.161)–(7.162).

7.4.6. *Use of the cone of convex functions*

The operator (7.151) transforms every non-negative integrable function $y(s)$ into a non-negative function $x(t)=Ay(t)$ which vanishes at the end-points of the interval [0, 1] and directed with convexity upward. In this connection, the function $x(t)$ will be continuously differentiable and will have a second derivative almost everywhere.

It is seen directly from the form of the operator A that it transforms every bounded set of functions in L_1 into a set which is compact with respect to a uniform norm. Moreover, the operator A transforms every almost everywhere converging sequence of fucntions which is bounded with respect to the norm in L_1 into a sequence of functions which converges uniformly.

We assume that the function $f(t, x, y)$ $(0\leqslant t\leqslant 1,\ x\geqslant 0,\ -\infty<y<\infty)$ is non-negative, continuous with respect to the collection of variables and uniformly bounded when x assumes values from a finite interval.

We consider the operator

$$Ty(t)=Af[t,\ y(t),\ y'(t)]$$

on the cone K of functions, which are continuous and convex upward on [0, 1] and satisfy the condition $y(0)=y(1)=0$. Direct verification shows that the operator T, defined on K, transforms K into itself and is completely continuous on K. Besides this, the fixed points of the operator T coincide with the solutions of the boundary value problem (7.161)–(7.162).

Various restrictions on the nature of the increase of the function $f(t, x, y)$ with respect to the variable x allow the application of various general principles from the preceding chapters.
For example, let

$$0 \leqslant f(t, x, y) \leqslant a + bx \qquad (0 \leqslant t \leqslant 1\,,\ x \geqslant 0\,,\ -\infty < y < \infty)\,.$$

Then the inequalities

$$\boldsymbol{T}y(t) \leqslant A[a + by(t)] \leqslant a_1 + b\|A\|\ \|y\|$$

will be valid for the operator $\boldsymbol{T}$ from which it follows that for $b\|A\| < 1$ the operator $\boldsymbol{T}$ leaves the intersection of the cone K with the sphere $\|x\| \leqslant R$ of sufficiently large radius R invariant. The existence of a fixed point then follows from Schauder's principle.
We further assume that $f(t, 0, 0) \equiv 0$ and raise the question about the existence of non-trivial fixed points.
Let a number $\delta_0 > 0$ and a number $b_1 > \pi^2$ exist such that

$$f(t, x, y) \geqslant b_1 x \qquad (0 \leqslant x \leqslant \delta_0)\,.$$

Then the inequality $\boldsymbol{T}y(t) \geqslant b_1 Ay(t)$ will be satisfied on the functions $y(t) \in K$ with a small norm, from which it follows that $\boldsymbol{T}y(t) \overline{\leqslant} y(t)$. Thus, we find ourselves in the conditions of the theorem about operators which compress the cone. The existence of a non-zero fixed point is proved.
The case when $f(t, x, y)$ contains "strong" non-linearities with respect to the variable x is more complicated.
For example, let numbers $\delta_0 > 0$ and $M_0 > 0$ exist such that

$$f(t, x, y) \geqslant a_0 x^{1+\varepsilon} \qquad (x \geqslant M_0\,,\ a_0 > 0)\,,$$
$$f(t, x, y) \leqslant a_1 x^{1+\varepsilon} \qquad (x \leqslant \delta_0\,,\ \ a_1 \geqslant 0)\,.$$

We denote by $h_0(t)$ the characteristic function of the segment $[\frac{1}{4}, \frac{3}{4}]$. For every function $y(t) \in K$, the inequality

$$y(t) \geqslant \frac{\|y\|}{4} h_0(t) = \frac{\max y(t)}{4} h_0(t)$$

evidently is satisfied.
The operator $\boldsymbol{T}$ satisfies the inequality

$$\boldsymbol{T}y(t) \leqslant a_1 A[y(t)]^{1+\varepsilon} \leqslant a_2 \|y\|^{1+\varepsilon}$$

on elements with a small norm from which it follows that $\boldsymbol{T}y(t) \overline{\geqslant} y(t)$. On elements with large norm,

$$Ty(t) \geqslant a_0 A \left\{ \left[\tfrac{1}{4} \|y\| \right]^{1+\varepsilon} h_0(t) \right\} = a_0 \left(\frac{\|y\|}{4} \right)^{1+\varepsilon} A h_0(t) ,$$

from which it follows that $Ty(t) \lesseqgtr y(t)$. Thus, the conditions of the principle of a non-zero fixed point for the operators expanding the cone are satisfied. This means that T has at least one non-zero fixed point.

With analogous arguments, the existence of multiple fixed points can be established if $f(t, x, y)$ has alternating parts of fast and slow increase with respect to the variable x. A detailed analysis of a similar situation is mentioned in section 7.6 for a more complicated equation in which the operator $-\ddot{x}$ is replaced by a Monge-Ampère operator. We note that we apply the same method for the case when the Monge-Ampère operator (Hessian) is considered for functions of many variables.

7.5. Periodic solutions of systems of the second order

7.5.1. *Odd solutions*

We consider the equation

$$\frac{d^2x}{dt^2} + f\left(t, x, \frac{dx}{dt}\right) = 0 , \tag{7.172}$$

where the function $f(t, u, v)$ is periodic with respect to the variable t. It will be convenient for us to assume that the period is equal to 2:

$$f(2+t, u, v) \equiv f(t, u, v) \qquad (-\infty < t, u, v < \infty) . \tag{7.173}$$

We assume that the function $f(t, u, v)$ satisfies the condition

$$f(-t, -u, v) \equiv -f(t, u, v) \quad (-\infty < t, u, v < \infty) . \tag{7.174}$$

In this case, for the construction of the periodical solutions of the equation (7.172) the following device can be applied.

We first look for the solution $x^*(t)$ of the equation (7.172) defined on $[0, 1]$ and satisfying the boundary conditions

$$x^*(0) = x^*(1) = 0 , \tag{7.175}$$

that is, we look for a solution of the two point problem (see section 7.4). Moreover, we define the function $x^*(t)$ on $[-1, 0]$ extending its definition by the equality

$$x^*(-t) = -x^*(t) . \tag{7.176}$$

It is not difficult to see that the function $x^*(t)$ will be continuously differ-

entiable on $[-1, 1]$. Finally, we extend the function $x^*(t)$ to all values as a periodic function with period $\omega=2$. It follows from the periodicity of $f(t, u, v)$ and from condition (7.174) that the periodic function $x^*(t)$ will satisfy the equation (7.182) for all numbers t.

The above arguments show that *under the conditions of Theorems* 7.22 *and* 7.23, *the equation* (7.172) *has periodic solutions if, in addition, identities* (7.173) *and* (7.174) *are satisfied.*

We consider as an example the equation

$$\frac{d^2x}{dt^2} + f(x) + h \sin \pi t = 0 , \tag{7.177}$$

where

$$f(x) = \begin{cases} -ax_0 + b(x+x_0), & \text{if } x \leqslant -x_0 , \\ ax, & \text{if } -x_0 \leqslant x \leqslant x_0 , \\ ax_0 + b(x-x_0), & \text{if } x_0 \leqslant x . \end{cases}$$

It is not difficult to see that for equation (7.177) the condition (7.166) is satisfied where $b_0(t)=b$. Therefore, the conditions of Theorem 7.22 will be satisfied if

$$b < \pi^2 . \tag{7.178}$$

The oddness of the function $f(x)$ allows the application of the consideration expressed above. Consequently, condition (7.178) is sufficient for the equation (7.177).

We can also apply the method described for the construction of periodic solutions for systems with many degrees of freedom.

7.5.2. *Autonomous systems*

The construction of periodic solutions for autonomous systems of differential equations is complicated since the period of the periodic solution sought is unknown a priori.

We consider the system

$$\begin{aligned} &\frac{d^2x_1}{dt^2} + f_1\left(x_1, \ldots, x_n, \frac{dx_1}{dt}, \ldots, \frac{dx_n}{dt}\right) = 0 , \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\frac{d^2x_n}{dt^2} + f_n\left(x_1, \ldots, x_n, \frac{dx_1}{dt}, \ldots, \frac{dx_n}{dt}\right) = 0 \end{aligned} \tag{7.179}$$

and we assume that it has a periodic solution with period 2ω. We then

make the substitution

$$t=\omega\tau\,. \tag{7.180}$$

System (7.179) assumes the form

$$\begin{aligned}
&\frac{d^2x_1}{d\tau^2}+\omega^2 f_1\left(x_1,\ldots,x_n,\frac{1}{\omega}\frac{dx_1}{d\tau},\ldots,\frac{1}{\omega}\frac{dx_n}{d\tau}\right)=0\,,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\frac{d^2x_n}{d\tau^2}+\omega^2 f_n\left(x_1,\ldots,x_n,\frac{1}{\omega}\frac{dx_1}{d\tau},\ldots,\frac{1}{\omega}\frac{dx_n}{d\tau}\right)=0\,.
\end{aligned} \tag{7.181}$$

This system has periodic solutions with period 2.

It is obvious that the converse statement is valid: if for some $\omega>0$ system (7.181) has a periodic solution with period 2, then the system (7.179) has a periodic solution with period 2ω. Thus, the problem about periodic solutions of an autonomous system (7.179) reduces to the question about the existence for some $\omega>0$ of periodic solutions for the systems (7.181) where periodic solutions are now sought with a given known period 2.

The system (7.181) depends in the simplest manner on a parameter ω when the functions f_i in the equations (7.179) do not depend on derivatives, that is, in the case when a system of the form

$$\begin{aligned}
&\frac{d^2x_1}{dt^2}+g_1(x_1,\ldots,x_n)=0\,,\\
&\ldots\ldots\ldots\ldots\ldots\\
&\frac{d^2x_n}{dt^2}+g_n(x_1,\ldots,x_n)=0
\end{aligned} \tag{7.182}$$

is considered.

We shall call such systems *frictionless systems*. After the substitution (7.180), frictionless systems reduce to the equations

$$\begin{aligned}
&\frac{d^2x_1}{d\tau^2}+\omega^2 g_1(x_1,\ldots,x_n)=0\,,\\
&\ldots\ldots\ldots\ldots\ldots\\
&\frac{d^2x_n}{d\tau^2}+\omega^2 g_n(x_1,\ldots,x_n)=0\,.
\end{aligned} \tag{7.183}$$

If the functions $f_i(x_1,\ldots,x_n,\ y_1,\ \ldots,y_n)$ in equations (7.179) or the functions $g_i(x_1,\ldots,x_n)$ in the equations (7.182) are odd with respect to the

collection of variables $x_1, \ldots, x_n$, then for the construction of periodic solutions the solutions of a two point problem can be constructed as a preliminary. In fact, if for all $i=1, 2, \ldots, n$

$$f_i(-x_1, -x_2, \ldots, -x_n, y_1, \ldots, y_n) \equiv -f_i(x_1, x_2, \ldots, x_n, y_1, \ldots, y_n) \tag{7.184}$$

or

$$g(-x_1, -x_2, \ldots, -x_n) \equiv -g(x_1, x_2, \ldots, x_n)\,, \tag{7.185}$$

then the solution

$$x^*(t) = \{x_1^*(t), \ldots, x_n^*(t)\} \tag{7.186}$$

of the system (7.179) or (7.182) is defined on [0, 1] and satisfies the conditions

$$x_i^*(0) = x_i^*(1) = 0 \qquad (i=1, \ldots, n)\,, \tag{7.187}$$

and can be extended to $[-1, 1]$ and after that by periodicity to all values of t.

7.5.3. *Continuum of periodic solutions for frictionless systems*
We consider the system of integral equations

$$\begin{aligned} x_1(\tau) &= \omega^2 \int_0^1 G(\tau, s) g_1[x_1(s), \ldots, x_n(s)]\, ds \equiv \omega^2 G_1 x(\tau)\,, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_n(\tau) &= \omega^2 \int_0^1 G(\tau, s) g_n[x_1(s), \ldots, x_n(s)]\, ds \equiv \omega^2 G_n x(\tau)\,, \end{aligned} \tag{7.188}$$

where $G(t, s)$ is Green's function, of the problem (7.150), defined by the formula (7.152).
We denote by C^n the space of continuous vector functions

$$x(\tau) = \{x_1(\tau), \ldots, x_n(\tau)\} \tag{7.189}$$

with the norm

$$\|x(\tau)\| = \max_{0 \leqslant \tau \leqslant 1} |x_1(\tau)| + \ldots + \max_{0 \leqslant \tau \leqslant 1} |x_n(\tau)|\,.$$

We denote the cone in the space C^n composed of vector functions (7.189) with non-negative components by K.
We consider the system (7.188) as the operator equation

$$Gx(\tau) = \frac{1}{\omega^2} x(\tau)\,, \tag{7.190}$$

where the non-linear completely continuous* operator G is defined by the equality

$$Gx(\tau)=\{G_1x(\tau),\ldots G_nx(\tau)\}\,. \tag{7.191}$$

In the sequel, it is assumed that the functions $g_i(x_1,\ldots,x_n)$ satisfy the conditions

$$g_i(x_1,\ldots,x_n)\geqslant 0 \qquad (x_j\geqslant 0,\ i,j=1,\ldots,n)\,. \tag{7.192}$$

Then the operator G leaves the cone K invariant and, for the proof of the existence of non-negative solutions for the system (7.188) and their investigation, the results of chapters 4–6 can be applied.

We assume that at least one function $g_{i_0}(x_1,\ldots,x_n)$ from among the functions $g_i(x_1,\ldots,x_n)(i=1,\ldots,n)$ satisfies the condition

$$g_{i_0}(x_1,\ldots,x_n)\geqslant kx_{i_0} \qquad (x_j\geqslant 0,\ j=1,\ldots,n)\,. \tag{7.193}$$

Then the operator G_-

$$G_-x(\tau)=\{0,\ldots,0,\ kAx_{i_0}(\tau),\ 0,\ldots,0\} \tag{7.194}$$

will be a minorant (see chapter 5) of the operator (7.191), that is, for an arbitrary vector function (7.189) the conditions

$$G_ix(\tau)\geqslant 0 \qquad (0\leqslant\tau\leqslant 1,\ i\neq i_0)$$

and

$$G_{i_0}x)\tau)\geqslant kAx_{i_0}(\tau) \qquad (0\leqslant\tau\leqslant 1)$$

will be satisfied.

The operator G_- satisfies the condition

$$G_-u^*(\tau)\geqslant\alpha u^*(\tau) \qquad (0\leqslant\tau\leqslant 1)\,, \tag{7.195}$$

where α is some positive number and for the vector function

$$u^*(\tau)=\{u_1^*(\tau),\ldots,u_n^*(\tau)\}$$

all the components $u_i^*(\tau)$ are equal to zero for $i\neq i_0$ and

$$u_{i_0}^*(\tau)=\tau(1-\tau) \qquad (0\leqslant\tau\leqslant 1)\,.$$

This fact follows from Lemma 7.6.

It follows from Theorem 5.7 that the operator A has an infinite continuous branch of characteristic vector functions in the cone K to which positive characteristic values correspond. This means that system (7.188) has a

* Here, and in the sequel, the functions g_i are assumed continuous with respect to the collection of variables.

continuum of solutions for some positive values where among these solutions there are vector functions with arbitrarily large norm.
The solutions of system (7.188) are evidently solutions of the system (7.183) and satisfy the zero boundary conditions (7.187). This implies the next theorem.

Theorem 7.24. *Let the functions* $g_i(x_1, \ldots, x_n)(i=1, \ldots, n)$ *be odd with respect to the collection of variables and non-negative for non-negative values of the independent variables. Let one of these functions satisfy the condition* (7.193).
Then the system (7.182) *has a continuum of distinct periodic solutions where among them there are solutions with amplitudes as large and as small as desired.*

7.5.4. *Odd solutions for systems with friction*

As an example, we prove one theorem about periodic solutions of the system (7.179) by the method of topological extension.
In this subsection, the case will be considered when the functions $f_i(x_1, \ldots, x_n, y_1, \ldots, y_n)$ are odd with respect to the collection of variables $x_1, \ldots, x_n$. Therefore (see subsection 7.5.2), for the proof of the existence of periodic solutions for system (7.179) it suffices to establish the existence for system (7.181) of solutions satisfying the zero boundary conditions (7.187).
We make (as in section 4) the substitution

$$-\frac{d^2x_i(\tau)}{d\tau^2} = v_i(\tau), \quad x_i(0)=x_i(1)=0 \qquad (i=1, \ldots, n) \tag{7.196}$$

in the equations (7.181). Obviously,

$$x_i(\tau)=\int_0^1 G(\tau, s)v_i(s)\,ds, \quad \frac{dx_i(\tau)}{d\tau} = \int_0^1 G_\tau'(\tau, s)v_i(s)\,ds, \tag{7.197}$$

and system (7.181) transforms into a system of non-linear integral equations

$$\begin{aligned} v_1(\tau)&=\omega^2 f_1\left[Av_1(\tau), \ldots, Av_n(\tau), \frac{1}{\omega}Bv_1(\tau), \ldots, \frac{1}{\omega}Bv_n(\tau)\right], \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ v_n(\tau)&=\omega^2 f_n\left[Av_1(\tau), \ldots, Av_n(\tau), \frac{1}{\omega}Bv_1(\tau), \ldots, \frac{1}{\omega}Bv_n(\tau)\right], \end{aligned} \tag{7.198}$$

where A and B are the operators (7.151) and (7.154):

$$Av_i(\tau)=\int_0^1 G(\tau, s)v_i(s)ds\,, \quad Bv_i(\tau)=\int_0^1 G_\tau'(\tau, s)v_i(s)ds\,. \tag{7.199}$$

We introduce for consideration the operators F_ω:

$$F_\omega v(\tau)=F_\omega\{v_1(\tau), \ldots, v_n(\tau)\}=\{F_\omega^{(1)}v(\tau), \ldots, F_\omega^{(1)}v(\tau)\}\,,$$

where

$$F_\omega^{(1)}v(\tau)=f_1\left[Av_1(\tau), \ldots, Av_n(\tau), \frac{1}{\omega}Bv_1(\tau), \ldots, \frac{1}{\omega}Bv_n(\tau)\right],$$

. . . . . .

$$F_\omega^{(n)}v(\tau)=f_n\left[Av_1(\tau), \ldots, Av_n(\tau), \frac{1}{\omega}Bv_1(\tau) \ \ldots, \frac{1}{\omega}Bv_n(\tau)\right].$$

We assume that the conditions

$$f_i(x_1, \ldots, x_n, y_1, \ldots, y_n)\geqslant 0 \quad (x_j\geqslant 0, \ -\infty<y_k<\infty, \ i,j,k=1, \ldots, n) \tag{7.200}$$

are satisfied and that at least one function f_{i_0} of the functions $f_1, \ldots, f_n$ satisfies the condition

$$f_{i_0}(x_1, \ldots, x_n, y_1, \ldots, y_n)\geqslant kx_{i_0} \quad (x_j\geqslant 0, \ -\infty<y_k<\infty, j, k=1, \ldots, n). \tag{7.201}$$

Then each of the operators F_ω will be completely continuous in C^n, leave the cone K invariant and have on K as a monotone minorant the linear operator (7.194).

From here, as in the proof of Theorem 7.24, we can deduce that every operator F_ω has an infinite continuous branch of characteristic vectors. Let

$$F_\omega v(\tau, \lambda)=\lambda v(\tau; \lambda)\,. \tag{7.202}$$

If we can succeed in proving that among those values λ for which the equation (7.202) has solutions, there is the value $\lambda=1/\omega^2$, then this solution will be a solution of system (7.198). With respect to this solution, solutions of system (7.181) can be constructed, using the formulas (7.197), satisfying the zero boundary conditions. If, in addition, the oddness conditions (7.184) are satisfied, then we obtain the theorem about the existence of periodic solutions for the system (7.179).

We assume that the inequality

$$\overline{\lim_{\|v(\tau;\lambda)\|\to 0}}\ \lambda< \underline{\lim_{\|v(\tau;\lambda)\|\to\infty}}\ \lambda \tag{7.203}$$

or the inequality

$$\underline{\lim}_{\|v(\tau;\lambda)\|\to 0} \lambda > \overline{\lim}_{\|v(\tau;\lambda)\|\to\infty} \lambda \tag{7.204}$$

has been established. Then, repeating the arguments cited for the proof of Lemma 5.3, we can show that to every value of λ contained between the bounds, indicated in (7.203) or in (7.204), there corresponds a characteristic vector of operator F_ω. From here the existence of a non-zero solution for the system of equations (7.198) follows if either

$$\overline{\lim}_{\|v(\tau;\lambda)\|\to 0} \lambda < \frac{1}{\omega^2} < \underline{\lim}_{\|v(\tau;\lambda)\|\to\infty} \lambda , \tag{7.205}$$

or

$$\underline{\lim}_{\|v(\tau;\lambda)\|\to 0} \lambda > \frac{1}{\omega^2} > \overline{\lim}_{\|v(\tau;\lambda)\|\to\infty} \lambda . \tag{7.206}$$

We shall give conditions when the inequalities (7.206) are realized. In this connection, we limit ourselves to the simplest case.
Let a number $\delta_0 > 0$ exist such that

$$f_i(x_1, \ldots, x_n, y_1, \ldots, y_n) \geqslant$$
$$\geqslant k_0(x_1 + \ldots + x_n)(0 \leqslant x_j \leqslant \delta_0, \ -\infty < y_n < \infty) . \tag{7.207}$$

Then the operator F_ω has a minorant $k_0 F_0$ in a neighborhood of the point θ, where

$$F_0 x(\tau) = \{A[x_1(\tau) + \ldots + x_n(\tau)], \ldots, A[x_1(\tau) + \ldots + x_n(\tau)]\} . \tag{7.208}$$

The operator F_0 is v_0-positive, where

$$v_0(\tau) = \{\tau(1-\tau), \ldots, \tau(1-\tau)\} ,$$

since by virtue of Lemma 7.6 the inequalities

$$\alpha\tau(1-\tau) \leqslant A[x_1(\tau) + \ldots + x_n(\tau)] \leqslant \beta\tau(1-\tau) \qquad (0 \leqslant \tau \leqslant 1)$$

will be satisfied where α and β are positive for an arbitrary non-zero vector function

$$x(\tau) = \{x_1(\tau), \ldots, x_n(\tau)\} .$$

It follows from the equalities (7.202) that for characteristic vector functions with small norms the inequalities

$$k_0 F_0 v(\tau; \lambda) \leqslant \lambda v(\tau; \lambda) \qquad (0 \leqslant \tau \leqslant 1)$$

are satisfied. Therefore, it follows from Theorems 2.17 and 2.18 that

$$\frac{\lambda}{k_0} \geqslant \lambda(F_0)\,, \tag{7.209}$$

where $\lambda(F_0)$ is a unique characteristic value of the operator (7.208) to which there corresponds a characteristic vector function from the cone k. In turn, the inequality

$$\varlimsup_{\|v(\tau;\,\lambda)\|\to 0} \lambda \geqslant k_0 \varkappa(F_0) \tag{7.210}$$

follows from (7.209).

We further assume that the inequalities

$$f_i(x_1, \ldots, x_n, y_1, \ldots, y_n) \leqslant k_1(x_1 + \ldots + x_n) + k_2 \qquad (x_j \geqslant 0,\ -\infty < y_k < \infty) \tag{7.211}$$

are satisfied. Then the operator F_ω satisfies the condition

$$F_\omega x(\tau) \overline{\leqslant} k_1 F_0 x(\tau) + w(\tau)$$

on the entire cone, where $w(t)$ is some fixed vector function. In particular,

$$\lambda v(\tau;\,\lambda) = F_\omega v(\tau;\,\lambda) \overline{\leqslant} k_1 F_0 v(\tau;\,\lambda) + w(\tau)\,. \tag{7.212}$$

The operator F_0 is v_0-monotone and therefore its spectrum lies in the circle $|\lambda| \leqslant \lambda(F_0)$. By virtue of Lemma 2.2, there can be introduced in C^n an equivalent norm $\|x(t)\|_0$ in which $\|F_0\| < \lambda(F_0) + \varepsilon$, where ε is an arbitrarily small number. By virtue of Lemma 2.3, this norm possesses the property of monotoneity since the usual norm in C^n possesses the property of monotoneity. It then follows from (7.212) that

$$\lambda \|v(\tau;\,\lambda)\|_0 \leqslant k_1[\lambda(F_0) + \varepsilon]\ \|v(\tau;\,\lambda)\|_0 + \|w(\tau)\|_0\,,$$

from which it follows that

$$\lambda \leqslant k_1[\lambda(F_0) + \varepsilon] + \frac{\|w(\tau)\|_0}{\|v(\tau;\,\lambda)\|_0}\,.$$

Therefore,

$$\varlimsup_{\|v(\tau;\,\lambda)\|\to\infty} \lambda = \varlimsup_{\|v(\tau;\,\lambda)\|_0\to\infty} \lambda \leqslant k_1[\lambda(F_0) + \varepsilon) + \varepsilon]\,,$$

and, since ε is arbitrary, then

$$\varlimsup_{\|v(\tau;\,\lambda)\|\to\infty} \lambda \leqslant k_1 \lambda(F_0)\,. \tag{7.213}$$

We assume that $k_1 < k_0$. It follows from (7.210) and (7.213) that the

inequalities (7.206) will be satisfied if

$$k_1\lambda(F_0) < \frac{1}{\omega^2} < k_0\lambda(F_0)\,. \tag{7.214}$$

The number $\lambda(F_0)$ appearing in the estimate (7.214) is not difficult to determine. To this end, we denote by

$$z(\tau)=\{z_1(\tau), \ldots, z_n(\tau)\}$$

the characteristic vector of the operator F_0 corresponding to the characteristic value $\lambda(F_0)$. Obviously

$$\lambda(F_0)z_1(\tau)=\int_0^1 G(\tau, s)[z_1(s)+ \ldots +z_n(s)]\,ds\,,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\lambda(F_0)z_n(\tau)=\int_0^1 G(\tau, s)[z_1(s)+ \ldots +z_n(s)]\,ds\,,$$

from which it follows that

$$\lambda(F_0)[z_1(\tau)+ \ldots +z_n(\tau)]=n\int_0^1 G(\tau, s)[z_1(s)+ \ldots +z_n(s)]\,ds\,.$$

This means that the function

$$z_0(\tau)=z_1(\tau)+ \ldots +z_n(\tau)$$

satisfies the equation

$$\frac{d^2z_0(\tau)}{dt^2} + \frac{n}{\lambda(F_0)}z_0(\tau)=0\,,$$

is non-negative on [0, 1] and satisfies the zero boundary conditions $z_0(0)=z_0(1)=0$. But then

$$z_0(\tau)=a\sin\pi\tau$$

and $\lambda(F_0)=n/\pi^2$. Therefore, the inequalities (7.214) can be rewritten in the form

$$\frac{nk_1}{\pi^2} < \frac{1}{\omega^2} < \frac{nk_0}{\pi^2}$$

or

$$\frac{\pi}{\sqrt{(nk_0)}} < \omega < \frac{\pi}{\sqrt{(nk_1)}}\,. \tag{7.215}$$

We have thus completed the proof.

Theorem 7.25. *Let conditions* (7.200), (7.201), (7.207) and (7.211) *be*

satisfied where $k_1 < k_0$. Let the functions $f_i(x_1, \ldots, x_n, y_1, \ldots, y_n)$ be odd with respect to the collection of variables $x_1, \ldots, x_n$.
Then the system

$$\frac{d^2x_1}{dt^2} + f_1\left(x_1, \ldots, x_n, \frac{dx_1}{dt}, \ldots, \frac{dx_n}{dt}\right) = 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{d^2x_n}{dt^2} + f_n\left(x_1, \ldots, x_n, \frac{dx_1}{dt}, \ldots, \frac{dx_n}{dt}\right) = 0$$

has non-trivial periodic solutions of all the periods 2ω satisfying the inequalities (7.215).

Theorems 7.24 and 7.25 can be supplemented by the conditions of uniqueness (concavity of the functions g_i), by the conditions of convergence of successive approximations, and so on.

7.5.5. *Positive periodic solutions of systems of general form*
If the functions $f_i(t, x_1, \ldots, x_n, y, \ldots, y_n)$ are periodic with respect to t, but do not possess the property of oddness or some other kind of symmetry with respect to the variables $x_1, \ldots, x_n, y_1, \ldots, y_n$, then the construction of periodic solutions from the solutions of the two point problem cannot be achieved successfully. However, in the general case, the transition to integral equations turns out to be useful.
We now explain a general plan of arguments by an example involving a single equation

$$\ddot{x} + f(t, x, \dot{x}) = 0 \tag{7.216}$$

with periodic functions $f(t, x, y)$ with respect to t (we let the period equal 1). We need to find solutions of this equation for the boundary conditions

$$x(0) = x(1), \quad \dot{x}(0) = \dot{x}(1). \tag{7.217}$$

We can pass to the integral equation for example by means of the substitution

$$-\ddot{x} + \alpha^2 x = v(t), \quad x(0) = x(1), \quad x(0) = \dot{x}(1), \tag{7.218}$$

where α is some positive number. It is not difficult to see that the substitution (7.218) reduces to the equality

$$x(t) = \int_0^1 G_\alpha(t, s) v(s)\, ds, \tag{7.219}$$

where

$$G_\alpha(t, s) = \frac{1}{2\alpha(e^\alpha - 1)} \begin{cases} e^{\alpha(t-s)} + e^{\alpha(1+s-t)}, & \text{if } s \leqslant t, \\ e^{\alpha(s-t)} + e^{\alpha(1+t-s)}, & \text{if } s \geqslant t. \end{cases} \tag{7.220}$$

Therefore, the problem concerning periodic solutions of the equation (7.216) turns out to be equivalent to the integral equation

$$v(t) = f\left[t, \int_0^1 G_\alpha(t, s)v(s)\,ds, \int_0^1 \frac{\partial}{\partial t} G_\alpha(t, s)v(d)\,ds\right] + \\ + \alpha \int_0^1 G_\alpha(t, s)v(s)\,ds. \tag{7.221}$$

For the study of the integral equation (7.221) the methods developed in the book can be applied. Here it is useful to consider the Green function (7.220) to be positive.

7.6. Dirichlet problem for a Monge-Ampère equation

7.6.1. *Normal mappings*

In this section, Ω denotes a bounded convex domain in the $\{x, y\}$ plane; Γ is the boundary of the domain Ω; $\bar{\Omega} = \Omega + \Gamma$.

Let the twice continuously differentiable function $z(x, y)$ be given on $\bar{\Omega}$. We consider the mapping

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y} \tag{7.222}$$

of the domain $\bar{\Omega}$ into the plane of variables $\{p, q\}$. The Jacobian of this transformation coincides with the value of the differential operator

$$\boldsymbol{H}(z) = rt - s^2, \tag{7.223}$$

where, as usual,

$$r = \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 z}{\partial x \partial y}, \quad t = \frac{\partial^2 z}{\partial y^2}.$$

If $rt - s^2 > 0$, then the mapping (7.222) will be a one-to-one mapping; in this connection, the function $z(x, y)$ will be strictly convex. We recall that a function is strictly convex if every plane of support to the graph of the function has with the graph of the function a unique common

point (in the case of smooth surfaces the planes of support are tangential planes).

We now consider an arbitrary convex surface over the domain $\bar{\Omega}$. We correspond to each point $\{x_0, y_0\}$ the collection of all points $\{p, q\}$ such that the planes

$$z - z(x_0, y_0) = p(x - x_0) + q(y - y_0)$$

are planes of support to the surfaces. On a convex surface there can be points of only three types: smooth, ridge, and conical. In the smooth points, a unique plane of support exists, that is, to smooth points in our correspondence there corresponds a unique point in the $\{p, q\}$ plane. To ridge points there corresponds a collection of planes of support passing through a common line; to such points there corresponds a segment or ray in the $\{p, q\}$ plane. Finally, to conical points there corresponds a convex figure with interior points in the $\{p, q\}$ plane. The mapping described is called *normal.*

We denote by $\Psi(\mathscr{E}; z)(\mathscr{E} \subset \bar{\Omega})$ the sum of normal images of all points $\{x, y\} \in \mathscr{E}$; the set $\Psi(\mathscr{E}; z)$ is called the normal image of the set $\mathscr{E}$.

We now consider the differential equation

$$H(x) \equiv rt - s^2 = \varphi(x, y) \tag{7.224}$$

with the non-negative function $\varphi(x, y)$. A solution $z(x, y)$ of this equation, if it exists, will be a convex function. This solution, obviously, will satisfy the equality

$$\int_{\mathscr{E}} \int (rt - s^2)\, dx\, dy = \int_{\mathscr{E}} \int \varphi(x, y)\, dx\, dy$$

for arbitrary measurable $\mathscr{E} \subset \bar{\Omega}$. The last equality can be written in the form

$$\int_{\mathscr{E}} \int \varphi(x, y)\, dx\, dy = \text{mes}\, \Psi(\mathscr{E}; z)\,. \tag{7.225}$$

The last equalities make sense for arbitrary (and not only smooth) convex functions $z(x, y)$. This leads to a generalization* of the concept of a solution of equation (7.224): the convex function $z(x, y)$ is a solution of the equation (7.224) if the equalities (7.225) are satisfied. Since the left member of the equalities (7.225) defines an absolutely continuous set

* This generalization goes back to A. D. Aleksandrov; our investigation follows the work of I. Ya. Bakel'man.

function, the solutions of the equation (7.224) do not have conical points.

7.6.2. *Auxiliary functions*

We consider the positive solutions of the differential equation

$$rt-s^2=R_0(1+p^2+q^2)^\alpha\,, \tag{7.226}$$

where $0\leqslant\alpha\leqslant 1$, satisfying a zero boundary condition on the boundary of the circle $x^2+y^2=r_0^2$. It is obvious that a surface of revolution for whose determination it is convenient to go over to polar coordinates r, θ will be the solution of the equation (7.226). The equations of meridians are written in the form

$$\frac{z_{rr}z_r}{(1+z_r^2)^\alpha}=R_0 r\,.$$

It is necessary to solve this equation on the interval $0\leqslant r\leqslant r_0$ under the conditions

$$z_r(0)=0\,,\quad z(r_0)=0\,.$$

Obviously

$$|z_r|^2\leqslant\begin{cases}[(1-\alpha)R_0r^2+1]^{1/(1-\alpha)}-1\,, & \text{if}\quad 0\leqslant\alpha<1\,,\\ e^{R_0r^2}-1\,, & \text{if}\quad \alpha=1\,,\end{cases}$$

from which it follows that

$$|z_r|\leqslant\begin{cases}\sqrt{[(1-\alpha)R_0r_0^2+1]^{1/(1-\alpha)}-1}\,, & \text{if}\quad 0\leqslant\alpha<1\,,\\ \sqrt{e^{R_0r_0^2}-1}\,, & \text{if}\quad \alpha=1\,,\end{cases} \tag{7.227}$$

and

$$\begin{cases}r_0\sqrt{[(1-\alpha)R_0r_0^2+1]^{1/(1-\alpha)}-1}\,, & \text{if}\quad 0\leqslant\alpha<1\,,\\ r_0\sqrt{e^{R_0r_0^2}-1}\,, & \text{if}\quad \alpha=1\,.\end{cases} \tag{7.228}$$

7.6.3. *Inversion of the Monge-Ampère operator*

The differential operator $(rt-s^2)(1+p^2+q^2)^{-\alpha}$, where $0\leqslant\alpha\leqslant 1$, will be called the *Monge-Ampère operator.*

We consider the boundary value problem

$$\frac{rt-s^2}{(1+p^2+q^2)^\alpha}=\varphi(x,y)\,,\quad z|_\Gamma=0\,. \tag{7.229}$$

For this boundary value problem the solution (analogous to subsection 7.6.1) is defined as a convex function satisfying the equalities

$$\int_{\mathscr{E}}\int \varphi(x, y)\,dx\,dy = \int_{\Psi(\mathscr{E};z)}\int (1+p^2+q^2)^{-\alpha}\,dp\,dq\,. \tag{7.230}$$

For the case of non-negative bounded (even summable) functions $\varphi(x, y)$, a solution of the problem (7.229) can be defined as a non-negative convex continuous function with an absolutely continuous square of the normal representation, satisfying the boundary condition and satisfying the differential equation almost everywhere.

It is known* that problem (7.229) has a unique non-negative solution $z(x, y)$. We call the operator A_α defined by the equality

$$A_\alpha\varphi(x, y)=z(x, y)$$

the *inversion of the Monge-Ampère operator.*

7.6.4. *Complete continuity of the operators* A_α

In addition, we assume that the specific curvature of the curve Γ is bounded below by a positive number $1/r_0$. This means that for every point $\{x_0, y_0\}\in\Gamma$ a circle $S(x_0, y_0)$ can be found of radius r_0, passing through this point, outside of which the domain Ω lies completely. Roughly speaking, this means that Γ does not contain rectilinear segments.

Without loss of generality, it can be assumed that $S(x_0, y_0)$ is the circle $x^2+y^2=r_0^2$. We denote the function constructed in subsection 7.6.2 by $u(x, y)$.

Let the function $\varphi(x, y)$ satisfy the inequality

$$0\leqslant\varphi(x, y)<R_0 \qquad (\{x, y\}\in\Omega)\,. \tag{7.231}$$

We show that the function $z(x, y)=A_\alpha\varphi$ then satisfies the inequality

$$z(x, y)\leqslant u(x, y) \qquad (\{x, y\}\in\Omega)\,. \tag{7.232}$$

We assume the contrary and denote by $\mathscr{E}$ one of the connectedness components of the set of those points in which the inequality

$$u(x, y)<z(x, y)$$

is satisfied. The normal image $\Psi(\mathscr{E}; z)$ of the set $\mathscr{E}$ will contain a normal

* For the case $\alpha=\frac{3}{2}$ (not entering into our considerations) this fact was established by A. D. Aleksandrov; for the case $\alpha=0$ by A. V. Pogorelov; for the general case, by I. Ya. Bakel'man.

image $\Psi(\mathscr{E}; u)$ of this same set. This follows from the fact that the "line" of intersection of the surfaces $z(x, y)$ and $u(x, y)$ above the set $\mathscr{E}$ is situated over every plane of support to the graph of the function $u(x, y)$ above $\mathscr{E}$. From the inclusion $\Psi(\mathscr{E}; u) \subset \Psi(\mathscr{E}; z)$ follows the inequality

$$\int_{\Psi(\mathscr{E};u)} \int \frac{dp\,dq}{(1+p^2+q^2)^{\alpha}} \leqslant \int_{\Psi(\mathscr{E};z)} \int \frac{dp\,dq}{(1+p^2+q^2)^{\alpha}} = \\ = \int_{\mathscr{E}} \int \varphi(x, y)\,dx\,dy < R_0 \text{ mes } \mathscr{E} .$$

On the other hand,

$$\int_{\Psi(\mathscr{E};u)} \int \frac{dp\,dq}{(1+p^2+q^2)^{\alpha}} = \int_{\mathscr{E}} \int \frac{u_{xx}u_{yy} - u_{xy}^2}{(1+u_x^2+u_y^2)^{\alpha}}\,dx\,dy = R_0 \text{ mes } \mathscr{E} .$$

We have arrived at a contradiction.

From (7.232) and (7.228) follows the estimate of the solution $z(x, y)$:

$$0 \leqslant z(x, y) \leqslant \\ \leqslant \begin{cases} r_0 \sqrt{[(1-\alpha) R_0 r_0^2 + 1]^{1/(1-\alpha)} - 1}\,, & \text{if } 0 \leqslant \alpha < 1\,, \\ r_0 \sqrt{e^{R_0 r_0^2} - 1}\,. & \text{if } \alpha = 1\,. \end{cases} \tag{7.233}$$

The normal image of the point $\{x_0, y_0\}$ relative to the function $z(x, y)$ is contained in a normal image of the function $u(x, y)$ which in turn is contained in the circle T whose radius is defined by the right member of inequality (7.227). Since $\{x_0, y_0\}$ is an arbitrary point of the boundary Γ of the domain Ω, then normal representations of all points of Γ are contained in the circle T. Inasmuch as the function $z(x, y)$ is convex, its normal representation lies in the same circle in which the normal representation of the boundary lies.

Thus, for inclinations of planes of support to the graph of the function $z(x, y)$ the estimate

$$p^2 + q^2 \leqslant \begin{cases} [(1-\alpha) R_0 r_0^2 + 1]^{1/(1-\alpha)} - 1\,, & \text{if } 0 \leqslant \alpha < 1\,, \\ e^{R_0 r_0^2} - 1\,, & \text{if } \alpha = 1 \end{cases} \tag{7.234}$$

is valid.

It follows from the last estimates that the operator A_α transforms every uniformly bounded set of non-negative functions into a compact set with respect to the distance function of the space C of non-negative convex functions having the value zero on Γ.

We now prove that the operator A_α transforms every pointwise almost

everywhere converging uniformly bounded sequence of functions into a sequence of convex functions which converges uniformly.

Assuming the contrary, we can find a uniformly bounded sequence $\varphi_n(x, y)$, converging pointwise to the function $\varphi_0(x, y)$ such that the sequence $z_n(x, y) = A_\alpha \varphi_n$ does not have a limit. Since the sequence $z_n(x, y)$ is compact, then, without loss of generality, it can be assumed that the sequences z_{2k+1} and z_{2k} $(k = 1, 2, \ldots)$ converge uniformly to distinct limits z^* and z^{**}. The functions $z^*(x, y)$ and $z^{**}(x, y)$ are convex. Then, as I. Ya. Bakel'man [2] has shown, for an arbitrary measurable $\mathscr{E} \subset \bar{\Omega}$ the equalities

$$\int_{\Psi(\mathscr{E}; z^*)} \int \frac{dp\,dq}{(1+p^2+q^2)^\alpha} = \int_{\Psi(\mathscr{E}; z^{**})} \int \frac{dp\,dq}{(1+p^2+q^2)^\alpha} = \\ = \int_{\mathscr{E}} \int \varphi_0(x, y)\,dx\,dy$$

hold. It follows from these equalities that $z^*(x, y) \equiv z^{**}(x, y)$. We have arrived at a contradiction.

Thus, *the operators A_α are completely continuous as operators acting from the cone of bounded non-negative functions (with point convergence) to the cone of non-negative convex continuous functions which vanish on Γ.*

7.6.5. *Further properties of the operators A_α*

We first note that the *operators A_α are monotone*: it follows from $0 \leqslant \varphi_1(x, y) \leqslant \varphi_2(x, y)$ that

$$0 \leqslant A_\alpha \varphi_1(x, y) \leqslant A_\alpha \varphi_2(x, y) \qquad (\{x, y\} \in \bar{\Omega}) . \tag{7.235}$$

It suffices to carry out the proof for the case when the strict inequalities

$$\varphi_1(x, y) < \varphi_2(x, y) \qquad (\{x, y\} \in \bar{\Omega})$$

hold, since, after this, the general case can be obtained by passage to the limit using the continuity of the operator A_α. If the inequalities (7.235) are not satisfied, then we denote by $\mathscr{E}$ one of the connectedness components of the set of points in which $A_\alpha \varphi_1(x, y) > A_\alpha \varphi_2(x, y)$; the normal image $\Psi(\mathscr{E}; A_\alpha \varphi_1)$ will contain the normal image $\Psi(\mathscr{E}; A_\alpha \varphi_2)$; therefore

$$\int_{\mathscr{E}} \int \varphi_2(x, y)\,dx\,dy = \int_{\Psi(\mathscr{E}; A_\alpha \varphi_2)} \int \frac{dp\,dq}{(1+p^2+q^2)^\alpha} \leqslant \\ \leqslant \int_{\Psi(\mathscr{E}; A_\alpha \varphi_1)} \int \frac{dp dq}{(1+p^2+q^2)^\alpha} = \int_{\mathscr{E}} \int \varphi_1(x, y)\,dx\,dy < \\ < \int_{\mathscr{E}} \int \varphi_2(x, y)\,dx\,dy ,$$

and we have arrived at a contradiction.

For $0 \leqslant \alpha < 1$ the operators A_α have the following property: *for every non-negative function φ the inequalities*

$$A_\alpha(\lambda\varphi) \geqslant \lambda^{1/(2(1-\alpha))} A^\alpha \varphi \qquad (0 \leqslant \lambda \leqslant 1) \tag{7.236}$$

are satisfied. For the proof, we introduce the notation $A_\alpha \varphi = u$ and assume first that the function $u(x, y)$ is sufficiently smooth. Obviously, for $0 < \mu \leqslant 1$

$$\left(\frac{1+u_x^2+u_y^2}{1+\mu^2 u_x^2+\mu^2 u_y^2}\right)^\alpha \leqslant \mu^{-2\alpha},$$

from which it follows that

$$\frac{\mu^2(u_{xx}u_{yy}, u_{xy}^2)}{(1+\mu^2 u_x^2+\mu^2 u_y^2)^\alpha} \leqslant \mu^{2(1-\alpha)} \frac{u_{xx}u_{yy}-u_{xy}^2}{(1+u_x^2+u_y^2)^\alpha} = \mu^{2(1-\alpha)}\varphi, \tag{7.237}$$

and it follows from the monotoneity of the operator A_α that

$$\mu u(x, y) \leqslant A_\alpha[\mu^{2(1-\alpha)}\varphi(x, y)]$$

or, what is the same, that

$$\mu A_\alpha \varphi \leqslant A_\alpha(\mu^{2(1-\alpha)}\varphi). \tag{7.238}$$

Performing in the last inequality the substitution $\mu^{2(1-\alpha)} = \lambda$, we arrive at (7.236).

In the case of an arbitrary bounded non-negative function $\varphi(x, y)$, the function $u(x, y)$ has a second differential* almost everywhere. Therefore the square of a normal image with weight $(1+u_x^2+u_y^2)^\alpha$ is reestablished with respect to the values $u_{xx}u_{yy}-u_{xy}^2/[(1+u_x^2+u_y^2)]^\alpha$ at those points where this expression is defined. At these points the inequality (7.237) is satisfied. Therefore, the function

$$\frac{\mu^2(u_{xx}u_{yy}-u_{xy}^2)}{(1+\mu^2 u_x^2+\mu^2 u_y^2)^\alpha} = \psi(x, y)$$

is bounded. The argument mentioned above shows that the square of a normal image for the function $A_\alpha\psi(x, y)$ also is absolutely continuous. Thus (7.238), and consequently (7.236), follows from (7.237).

In the sequel, one further fact will be used: *if $\alpha_1 < \alpha_2$ then for every non-negative function $\varphi(x, y)$ the inequality*

$$A_{\alpha_1}\varphi(x, y) \leqslant A_{\alpha_2}\varphi(x, y) \qquad (\{x, y\} \in \bar{\Omega}) \tag{7.239}$$

* See Busemann and Feller [1], A. D. Aleksandrov [2].

is satisfied. For the proof, we introduce the notation

$$v(x, y)=A_{\alpha_2}\varphi(x, y)\,.$$

Then

$$\frac{v_{xx}v_{yy}-v_{xy}^2}{(1+v_x^2+v_y^2)^{\alpha_1}}=\varphi(x, y)(1+v_x^2+v_y^2)^{\alpha_2-\alpha_1}\geqslant\varphi(x, y)$$

and (7.239) follows from the monotoneity of the operator A_{α_1}.

7.6.6. *Existence of solutions*

We study the question about the existence of solutions for the equation

$$\frac{rt-s^2}{(1+p^2+q^2)^{\alpha}}=f(x, y, z, p, q)\,, \tag{7.240}$$

satisfying the conditions

$$z(x, y)\geqslant 0,\quad z|_{\Gamma}=0\,. \tag{7.241}$$

Everywhere in the sequel, it is assumed that $f(x, y, z, p, q)(\{x, y,\}\in\Omega, \bar{z}\geqslant 0, -\infty<p, q<\infty)$ is continuous with respect to the collection of variables, is non-negative and uniformly bounded if the values of z belong to a finite interval.

Theorem 7.26. *Let the condition*

$$0\leqslant f(x, y, z, p, q)\leqslant\begin{cases}a_1(1+z^2)^{\gamma}\,, & \text{if}\quad 0\leqslant\alpha<1\,,\\ a_1\ln^{1-\varepsilon}(2+z)\,, & \text{if}\quad \alpha=1\end{cases} \tag{7.242}$$

be satisfied, where $\gamma<1-\alpha$, $\varepsilon>0$, $a_1>0$.
Then the boundary value problem (7.240)–(7.241) *has at least one solution.*

Proof. We consider the operator

$$B\varphi(x, y)=A_{\alpha}f[x, y, \varphi(x, y),\ \varphi_x'(x, y),\ \varphi_y'(x, y)]\,. \tag{7.243}$$

If the function $\varphi(x, y)$ is non-negative, convex and satisfies conditon (7.241), then the derivatives $\varphi_x'(x, y)$ and $\varphi_y'(x, y)$ exist almost everywhere. Therefore, the function

$$\psi(x, y)=f\varphi(x, y)=f[x, y, \varphi(x, y),\ \varphi_x'(x, y),\ \varphi_y'(x, y)] \tag{7.244}$$

is defined almost everywhere and is bounded. It follows from the properties of the operator A_{α} that the function $A_{\alpha}\psi(x, y)$ is defined. Thus, the operator B is defined on the cone K of non-negative convex continuous functions which vanish on Γ. The values of the operator B belong to the

same cone which everywhere in the sequel is considered as a cone in the space C of continuous functions.

We denote by K_R the intersection of the cone K with the sphere $\|\varphi\| \leqslant R$. It follows from the estimate (7.242) that the functions $f\varphi(x, y) (\varphi \subset K_R)$ are uniformly bounded. Therefore, the set $\boldsymbol{B}K_R$ is compact in C, that is, the operator $\boldsymbol{B}$ is compact on the cone K. Now let the sequence $\varphi_n(x, y)$ converge to $\varphi_0(x, y)$. As is known,* the functions $(\partial/\partial x)\, \varphi_n(x, y)$ and $(\partial/\partial y)\, \varphi_n(x, y)$ converge almost everywhere to $(\partial/\partial x)\, \varphi_0(x, y)$ and $(\partial/\partial y)\, \varphi_0(x, y)$. Therefore, the uniformly bounded sequence of functions $f\varphi_n(x, y)$ converges pointwise to $f\varphi_0(x, y)$. But then the sequence $A_\alpha f\varphi_n(x, y)$ converges uniformly to $A_\alpha f\varphi_0(x, y)$.

We have shown that the operator $\boldsymbol{B}$ is completely continuous on the cone K. It is not difficult to see that the fixed points of the operator $\boldsymbol{B}$ coincide with the solutions of problem (7.240)–(7.241).

The statement of the theorem will follow from Schauder's principle if it is shown that for some number R the inclusion

$$\boldsymbol{B}K_R \subset K_R \tag{7.245}$$

holds.

First, let $0 \leqslant \alpha < 1$. Then it follows from (7.242) that

$$f\varphi(x, y) \leqslant a_1(1 + R^2)^\nu \qquad (\varphi(x, y) \in K_R)\,,$$

from which by virtue of (7.233)

$$\boldsymbol{B}\varphi(x, y) \leqslant r_0 \sqrt{[(1-\alpha)a_1(1+R^2)^\nu r_0^2 + 1]^{1/(1-\alpha)} - 1} \qquad (\varphi(x, y) \in K_R)\,.$$

It follows from this inequality that (7.245) is satisfied for large numbers R.

Now let $\alpha = 1$. It then follows from (7.242) and (7.233) that

$$\boldsymbol{B}\varphi(x\ y) \leqslant r_0 \sqrt{e^{a_1 r_0^2 \ln^{1-\varepsilon}(2+R)} - 1} \quad (\varphi(x, y) \in K_R)\,. \tag{7.246}$$

It follows from the inequalities $uv \leqslant \varepsilon u^{1/\varepsilon} + (1-\varepsilon)v^{1/(1-\varepsilon)}$, that for an arbitrary number $\delta > 0$

$$\ln^{1-\varepsilon}(2+R) \leqslant \varepsilon\delta^{-1/\varepsilon} + (1-\varepsilon)\delta^{1/(1-\varepsilon)} \ln(2+R)\,.$$

Therefore, (7.246) implies the inequality

$$\boldsymbol{B}\varphi(x, y) \leqslant r_0 \sqrt{e^{\varepsilon\delta^{-1/\varepsilon} a_1 r_0^2 (2+R)^{(1-\varepsilon)\delta^{1/(1-\varepsilon)} a_1 r_0^2}} - 1}\,.$$

* See A. D. Aleksandrov [2].

If the number δ is chosen sufficiently small, then a large number R can be chosen for which the inequality

$$B\varphi(x, y) < R \qquad (\varphi(x, y) \in K_R) \tag{7.247}$$

is satisfied and from which (7.245) follows.
This completes the proof of the theorem.

7.6.7. *Second solution*
We further assume that

$$f(x, y, 0, 0, 0) \equiv 0 \qquad (\{x, y\} \in \bar{\Omega}) . \tag{7.248}$$

Then the boundary value problem (7.240)–(7.241) has the trivial zero solution.

Theorem 7.27. *Let the conditions of Theorem* 7.26 *and condition* (7.248) *be satisfied and let a number* $\delta_0 > 0$ *be found such that*

$$f(x, y, z, p, q) \geqslant a_2 z^{2-\varepsilon} \qquad (0 \leqslant z \leqslant \delta_0, \ -\infty < p, q < \infty) , \tag{7.249}$$

where $a_2 > 0$.
Then the problem (7.240)–(7.241) *has at least one solution besides the trivial solution.*

Proof. Let S_1, S_2 and S_3 be three concentric circles of radii r_1, r_2 and r_3 respectively, the first two of which lie in the domain Ω and the third contains $\bar{\Omega}$ in its interior. We shall denote by $h_0(x, y)$ the characteristic function of the circle S_1.
We consider an arbitrary convex function $\varphi(x, y)$ on Γ, which vanishes on Ω. The inequality

$$\varphi(x, y) \geqslant \frac{r_2 - r_1}{r_2 + r_3} \|\varphi\| h_0(x, y) \qquad (\{x \ y\} \in \bar{\Omega}) \tag{7.250}$$

holds of which it is easy to convince ourselves by considering the graphs of the function $\varphi(x, y)$ in vertical planes passing through the common center of the circles S_1, S_2 and S_3 (Fig. 6).
The inequality (7.247) is satisfied under the conditions of Theorem 7.26 for large numbers R. If we show that for non-zero convex functions $\varphi(x, y)$ with a sufficiently small norm the inequality

$$B\varphi(x, y) \leqq \varphi(x, y) \tag{7.251}$$

is satisfied where B is the operator (7.243), then from the theorems about operators compressing the cone, it will follow that B has a non-zero fixed point in the cone K. With this the proof will be completed.

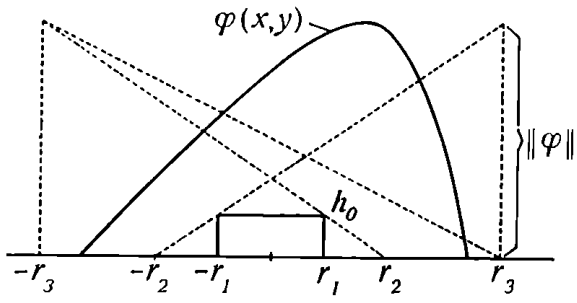


Fig. 6

We remark that the relation (7.251) can be thought of in the sense of a partial ordering generated by the cone K of convex functions or in the sense of a partial ordering generated by the cone of all non-negative functions on $\bar{\Omega}$.

It follows from (7.249) that with a small norm

$$f\varphi(x, y) \geqslant a_2[\varphi(x, y)]^{2-\varepsilon}$$

for functions $\varphi(x, y)$ with a small norm and, by virtue of (7.250),

$$f\varphi(x\, y) \geqslant a_2\left(\frac{r_2-r_1}{r_2+r_3}\,\|\varphi\|\right)^{2-\varepsilon} h_0(x, y)\,.$$

Moreover, it follows from (7.239) and from the monotoneity of the operator A_0 that

$$A_\alpha f\varphi \geqslant A_0 f\varphi \geqslant A_0\left[\alpha_2\left(\frac{r_2-r_1}{r_2+r_3}\,\|\varphi\|\right)^{2-\varepsilon} h_0(x, y)\right]. \tag{7.252}$$

If the norm of the function $\varphi(x, y)$ is sufficiently small, then the inequality

$$a_2\left(\frac{r_2-r_1}{r_2+r_3}\,\|\varphi\|\right)^{2-\varepsilon} < 1$$

holds, and by virtue of (7.236)

$$A_0\left\{a_2\left(\frac{r_2-r_1}{r_2+r_3}\,\|\varphi\|\right)^{2-\varepsilon} h_0(x, y)\right\} \geqslant \sqrt{a_2}\left(\frac{r_2-r_1}{r_2+r_3}\,\|\varphi\|\right)^{1-(\varepsilon/2)} A_0 h_0(x, y)\,.$$

Combining the last inequality with (7.252), we arrive at the relation

$$B\varphi(x, y) \geqslant \|\varphi\|^{1-(\varepsilon/2)}\, w_0(x, y)\,, \tag{7.253}$$

where

$$w_0(x, y)=\sqrt{a_2}\left(\frac{r_2-r_1}{r_2+r_3}\right)^{1-(\varepsilon/2)} A_0 h_0(x, y)\,.$$

It follows directly from (7.253) that for convex functions with a small non-zero norm the condition (7.251) is satisfied.
This completes the proof of the theorem.

7.6.8. *Uniqueness of a non-zero solution*
We consider a particular case of problem (7.240)–(7.241):

$$\frac{rt-s^2}{(1+p^2+q^2)^\alpha}=f(x, y, z), \quad z|_\Gamma=0 \tag{7.254}$$

where $0\leqslant\alpha<1$. Let the theorem on the existence of a non-zero non-negative solution (for example, the conditions of Theorem 7.27 are satisfied) be valid for the problem (7.254). In this subsection we shall give sufficient conditions for the uniqueness of such a solution.
We shall say that the function $f(x, y, z)$ is *quasi-homogeneous of degree* γ, *if* $f(x, y, z)$ does not decrease with respect to z, if

$$f(x, y, z)>0 \qquad (\{x, y\}\in\Omega\,, \quad z>0) \tag{7.255}$$

and if

$$f(x, y, \lambda z)\geqslant\lambda^\gamma f(x, y, z) \qquad (\{x, y\}\in\bar{\Omega}, \quad 0\leqslant\lambda\leqslant 1\,, \quad z\geqslant 0)\,. \tag{7.256}$$

The function $a+bz^\gamma$ $(a, b\geqslant 0)$ can serve as an example.

Theorem 7.28. *Let* $f(x, y, z)$ *be quasi-homogeneous of degree* γ, *where*

$$\gamma<2(1-\alpha)\,. \tag{7.257}$$

Then the boundary value problem (7.254) *does not have more than one non-zero non-negative solution.*

Proof. As in the proof of the two preceding theorems, we go over from the boundary value problem (7.254) to the equation $\varphi=B\varphi$, where

$$B\varphi(x, y)=A_\alpha f[x, y, \varphi(x, y)]\,. \tag{7.258}$$

It is necessary for us to prove that, under the conditions of the theorem being proved, the operator B has not more than one non-zero fixed point in the cone K. To this end, it suffices to establish that the operator B is a u_0-concave operator on the cone K with respect to the partial ordering

defined by the cone of all non-negative continuous functions (then Theorem 6.3 can be applied).

We choose as $u_0(x, y)$ a function whose graph is a cone of unit height with vertex which projects into some fixed interior point $\{x_0, y_0\}$ of the domain Ω and with directrix Γ.

Let $\varphi(x, y)$ be a non-zero function of K. Then $z(x, y) = B\varphi(x, y)$ is also a non-zero function of K. Geometrically, it is obvious that

$$z(x_0, y_0)u_0(x, y) \leqslant z(x, y) = B\varphi(x, y) \qquad (\{x, y\} \in \bar{\Omega}) .$$

It follows from the estimates obtained in subsection 7.6.3 that the inclinations of the planes of support to the graph of function $z(x, y)$ are uniformly bounded above by some constant L. Inclinations of the planes of support to graphs of functions $\beta u_0(x, y)$ at the points of the contour Γ are bounded below by some constant which increases indefinitely as $\beta \to 0$. Therefore, a number β_0 can be found such that the inclinations of the planes of support at the points of the contour Γ to the graph of function $\beta_0 u_0(x, y)$ will be greater than L. But then (the graph of $\beta_0 u_0(x, y)$ is a cone!) the inequality

$$B\varphi(x, y) = z(x, y) \leqslant \beta_0 u_0(x, y)$$

will be satisfied. We have shown that

$$z(x_0, y_0)u_0(x, y^+ \leqslant B\varphi(x, y) \leqslant \beta_0 u_0(x, y) ,$$

that is, condition (6.1) in the definition of the concavity of an operator is satisfied.

The monotoneity of the operator B follows from the monotoneity of $f(x, y, z)$ with respect to z and from the monotoneity of the operator A_α.

It remains to prove the property (6.5), that is, to prove that for $0 < \lambda < 1$ and an arbitrary function $\varphi \in K$ different from zero,

$$B(\lambda\varphi) \geqslant \lambda(1 + \eta)B\varphi , \tag{7.259}$$

where $\eta > 0$. In fact, it follows from (7.256) that

$$B[\lambda\varphi(x, y)] \geqslant A_\alpha\{\lambda^\nu f[x, y, z(x, y)]\}$$

and, by virtue of (7.236),

$$B[\lambda\varphi(x, y)] \geqslant \lambda^{[2(1-\alpha)]/\nu} B\varphi(x, y) \qquad (\{x, y\} \in \bar{\Omega}) .$$

The last inequality means that (7.259) is satisfied for

$$\eta = \lambda^{-[2(1-\alpha)-\nu]/(1-\alpha)} - 1 .$$

This completes the proof of the theorem.

7.6.9. *Equations with strong non-linearities*

Theorem 7.29. *Let numbers $\delta_0>0$ and $M_0>0$ be found such that*

$$f(x, y, z, p, q)\leqslant a_3 z^\nu \tag{7.260}$$
$$(\{x, y\}\in\bar{\Omega}\,,\quad 0\leqslant z\leqslant\delta_0\,,\quad -\infty<p, q<\infty)$$

and

$$f(x, y, z, p, q)\geqslant a_4 z^\nu \tag{7.261}$$
$$(\{x, y\}\in\bar{\Omega}\,,\quad z\geqslant M_0\,,\quad -\infty<p, q<\infty)\,,$$

where $\nu>2$, $a_4>0$.

Then the boundary value problem (7.240)–(7.241) *has besides the trivial zero solution at least one more solution.*

Proof. We turn from the boundary value problem to the operator equation $\varphi(x, y)=\boldsymbol{B}\varphi(x, y)$, where $\boldsymbol{B}$ is the operator (7.243), and we show that the operator $\boldsymbol{B}$ is an expansion of the cone K. With this the proof will be completed.

We first show that for sufficiently large numbers R the relation

$$\boldsymbol{B}\varphi(x, y)\overline{\leqslant}\varphi(x, y)\qquad(\varphi\in K,\ \|\varphi\|=R) \tag{7.262}$$

is satisfied.

In fact from inequality (7.250) follows the inequality

$$f[x, y, \varphi(x, y),\ \varphi_x'(x, y),\ \varphi_y'(x, y)]\geqslant a_4\left(\frac{r_2-r_1}{r_2+r_3}\,R\right)^\nu h_0(x, y)$$

provided

$$\frac{r_2-r_1}{r_2+r_3}\,R\geqslant M_0\,.$$

Therefore,

$$\boldsymbol{B}\varphi(x, y)\geqslant \boldsymbol{A}_\alpha\left[a_2\left(\frac{r_1-r_2}{r_2+r_3}\,R\right)^\nu h_0(x, y)\right]$$

and, by virtue of (7.239),

$$\boldsymbol{B}\varphi(x\ y)\geqslant A_0\left[a_4\left(\frac{r_2-r_1}{r_2+r_3}\,R\right)^\nu h_0(x, y)\right]\ =\ \sqrt{a_4}\left(\frac{r_2-r_1}{r_2+r_3}\right)^{\nu/2} R^{\nu/2} A_0 h_0(x, y).$$

It follows from the inequality just obtained that (7.262) is satisfied for sufficiently large numbers R.

It remains to show that for sufficiently small positive numbers ϱ the relation

$$B\varphi(x, y) \overline{\leqslant} \varphi(x, y) \qquad (\varphi \in K,\ \|\varphi\| = \varrho) \tag{7.263}$$

is satisfied. The proof of this relation is given separately for the cases $0 \leqslant \alpha < 1$ and $\alpha = 1$.

Let $0 \leqslant \alpha < 1$. Then for $\varrho < \delta_0$

$$f[x, y, \varphi(x, y),\ \varphi_x'(x, y),\ \varphi_y'(x, y)] \leqslant a_3 \varrho^{\gamma} \qquad (\varphi \in K,\ \|\varphi\| = \varrho),$$

and it follows from the estimates (7.233) that

$$B\varphi(x, y) \leqslant r_0 \sqrt{[(1-\alpha) a_3 \varrho^{\gamma} r_0^2 + 1]^{1/(1-\alpha)} - 1},$$

from which follows the validity of the relations (7.263) for small numbers ϱ.

Let $\alpha = 1$. Then

$$B\varphi(x, y) \leqslant r_0 \sqrt{e^{a_3 \varrho^{\gamma} r_0^2} - 1}$$

and again for small numbers ϱ

$$B\varphi(x, y) < \varrho .$$

This completes the proof of the theorem.

7.6.10. *Conditions for the existence of many solutions*

Let $h_0(x, y)$ be the function introduced in the proof of Theorem 7.27 and let $Ah_0(x, y) = g_0(x, y)$ and

$$d_0 = \max A_0 h_0(x, y) = g_0(x_0, y_0) .$$

In the case when the function $f(x, y, z, p, q)$ has, with respect to the variable z, alternating parts of fast growth and parts of slow growth, the boundary value problem (7.240)–(7.241) has, generally speaking, many solutions. We present one partial theorem for the case when $0 \leqslant \alpha < 1$.

Theorem 7.30. *Let a sequence $R_n \to \infty$ exist such that*

$$f(x, y, z, p, q) \geqslant a z^{\gamma_1} \qquad \left(\frac{r_2 - r_1}{r_2 + r_3} R_n \leqslant z \leqslant R_n\right), \tag{7.264}$$

where $\gamma_1 > 2$. Further, let a sequence $R_n^* \to \infty$ exist such that

$$f(x, y, z, p, q) \leqslant a_n (1 + z^2)^{\gamma_2} \qquad (0 \leqslant z \leqslant R_n^*), \tag{7.265}$$

where $\gamma_2 < 1 - \alpha$ and the growth of the numbers a_n agrees with the growth of the numbers R_n^* by the inequalities

$$r_0 \sqrt{[(1-\alpha) a_n (1 + R_n^{*2})^{\gamma_2} r_0^2 + 1]^{1/(1-\alpha)} - 1} < R_n^* ,$$

in which $1/r_0$ is the lower estimate for the specific curvature of the contour Γ. Then the boundary value problem (7.240)–(7.241) *has a denumerable number of distinct solutions among which there is a solution with a maximum as large as desired.*

Proof. Without loss of generality, it can be assumed that the numbers R_n satisfy the inequalities

$$R_n > \left(\frac{1}{ad_0^2}\right)^{1/(\gamma_1-2)} \left(\frac{r_2+r_3}{r_2-r_1}\right)^{\gamma_1/(\gamma_1-2)}.$$

Then for every function $\varphi(x, y)\in K$, $\|\varphi\| = R_n$ implies that

$$\varphi(x, y) \geqslant \frac{r_2-r_1}{r_2+r_3} R_n h_0(x, y),$$

from which it follows that

$$\begin{aligned} \boldsymbol{B}\varphi(x, y) &\geqslant A_\alpha\left[a\left(\frac{r_2-r_1}{r_2+r_3} R_n\right)^{\gamma_1} h_0(x, y)\right] \geqslant \\ &\geqslant A_0\left[a\left(\frac{r_2-r_1}{r_2+r_3} R_n\right)^{\gamma_1} h_0(x, y)\right] = \\ &= \sqrt{a}\left(\frac{r_2-r_1}{r_2+r_3} R_n\right)^{\gamma_1/2} A_0 h_0(x, y) \end{aligned}$$

and

$$\boldsymbol{B}\varphi(x_0, y_0) \geqslant \sqrt{ad_0^2}\left(\frac{r_2-r_1}{r_2+r_3} R_n\right)^{\gamma_1/2} > R_n.$$

Thus, it follows from condition (7.264) that

$$\boldsymbol{B}\varphi(x, y) \lesseqgtr \varphi(x, y) \qquad (\varphi\in K,\ \|\varphi\| = R_n). \tag{7.266}$$

It follows from condition (7.265) that

$$\boldsymbol{B}\varphi(x, y) \gtreqless \varphi(x, y) \qquad (\varphi\in K,\ \|\varphi\| = R_n^*). \tag{7.267}$$

In order to convince ourselves of this, it suffices to verify the corresponding part of the proof of Theorem 7.26.

Without loss of generality, it can be assumed that the numbers R_n and R_n^* alternate (otherwise, we would pass to a subsequence):

$$R_1 < R_1^* < R_2 < R_2^* < \ldots$$

Then the operator $\boldsymbol{B}$ is an operator of compression on every shell $R_n \leqslant \|\varphi\| \leqslant R_n^* (\varphi\in K)$ and is an operator of expansion on every shell $R_n^* \leqslant \|\varphi\| \leqslant R_{n+1} (\varphi\in K)$. Therefore, in all the indicated shells the operator $\boldsymbol{B}$ has at least one fixed point.

It remains to use the fact that the fixed points of the operator $\boldsymbol{B}$ coincide with the solutions of the boundary value problem (7.240)–(7.241). This completes the proof of the theorem.

The theorem remains valid for the case when $\alpha = 1$ if the condition (7.265) is replaced by the condition

$$f(x, y, z, p, q) \leqslant a_n \ln^{1-\varepsilon}(2+z) \qquad (0 \leqslant z \leqslant R_n^*), \tag{7.268}$$

where the growth of the numbers a_n agrees with the growth of the numbers R_n^* by the inequalities

$$r_0 \sqrt{e^{a_n r_0^2 \ln^{1-\varepsilon}(2+R_n^*)} - 1} < R_n^* .$$

7.6.11. *Remarks*

(1) The theorems proved in this section allow generalization to the case of equations

$$\frac{rt - s^2}{R(p, q)} = f(x, y, z, p, q),$$

where $R(p, q)$ is different from $(1+p^2+q^2)^\alpha$ $(0 \leqslant \alpha \leqslant 1)$. Here, one should keep in mind only that the admissible values of $f(x, y, z, p, q)$ can be bounded above if

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dp\,dq}{R(p, q)} < \infty .$$

(2) I. Ya. Bakel'man established theorems from which it follows that the operator

$$A_\alpha[E(x, y, z, p, q)r + F(x, y, z, p, q)s + G(x, y, z, p, q)t]$$

is completely continuous on the cone K (of convex functions) and leaves this cone invariant if

$$0 \leqslant -E(x, y, z, p, q)\xi^2 - F(x, y, z, p, q)\xi\eta - \\ - G(x, y, z, p, q)\eta^2 \leqslant \frac{\xi^2+\eta^2}{Q_1(p)+Q_2(q)},$$

where $Q_1(p)$ and $Q_2(q)$ are non-negative and

$$\int_{-\infty}^{\infty} \frac{dp}{Q_1(p)} < \infty , \quad \int_{-\infty}^{\infty} \frac{dq}{Q_2(q)} < \infty .$$

Therefore, the theorems proved in this section allow a generalization to the case of the equations

$$\frac{rt-s^2}{(1+p^2+q^2)^\alpha} = E(x, y, z, p, q)r+F(x, y, z, p, qp)s+ \\ +G(x, y, z, p, q)t+f(x, y, z, p, q)$$

with a quasi-linear part.

SUMMARY OF FUNDAMENTAL RESULTS (CHAPTERS 1-7)

Chapter 1. Spaces with a cone

1.1. *Basic definitions*

The set K forms, by definition, a *cone* in a real Banach space E provided it is convex and closed and if together with every element x it contains all the elements $\alpha x (\alpha \geqslant 0)$ and if it follows from x, $-x \in K$ that $x = \theta$, where θ is the zero of the space E.

The collections K_+ of non-negative functions in the spaces C, L_p and so on are the basic example of cones.

The cone K is called *solid* if it contains interior elements; the cone K is *reproducing* if every element $x \in E$ is representable in the form $x = u - v$, where $u, v \in K$.

The cone of non-negative functions is solid in the space C but in the spaces L_p it is not solid but reproducing.

The space E with the cone K is transformed into a *partially ordered space* if it is assumed that $x \leqslant y$ if $y - x \in K$. The sign $\leqslant$ possesses the property of the common sign $\leqslant$: the inequalities can be added and multiplied by positive numbers; it follows from $x \leqslant y$ and $y \leqslant z$ that $x \leqslant z$, we can pass in the inequalities to a limit, and so on.

1.2. *Normal cones*

Most often cones which have additional properties must be utilized. The cone K is called *normal* if there exists a number $\delta > 0$ such that $\|e_1 + e_2\| > \delta$ if $e_1 \in K$, $e_2 \in K$ and $\|e_1\| = \|e_2\| = 1$. The cones of non-negative functions are normal in the spaces C and L_p.

The norm in E is called *semi-monotone* if for arbitrary $x, y \in K$ it follows from $x \leqslant y$ that $\|x\| \leqslant N\|y\|$, where the constant N does not depend on x and y.

Theorem (1.2). *A necessary and sufficient condition for the cone K to be normal is that the norm be semi-monotone.*

Let u_0 be a fixed non-zero element of K. We denote by E_{u_0} the collection of all $x \in E$, which for some positive numbers t satisfy the inequalities $-tu_0 \leqslant x \leqslant tu_0$. The least number t for which these inequalities are satisfied is called the *u_0-norm* of the element $x \in E_{u_0}$ and is denoted by $\|x\|_{u_0}$.

Theorem (1.1). *A necessary and sufficient condition for the cone K to be normal is that the inequality*

$$\|x\| \leqslant M\|x\|_{u_0} \cdot \|u_0\| \qquad (x \in E_{u_0})$$

hold, where M does not depend on x or on u_0.

1.3. *The space E_{u_0}*

If E is the space L_p, K is the cone of non-negative functions, and $u_0(t) \equiv 1$, then E_{u_0} is the space L_p.

Theorem (1.3). *If the cone K is normal, then E_{u_0} is complete with respect to the u_0-norm.*

We denote the intersection $K \cap E_{u_0}$ by K_{u_0}. If E_{u_0} is complete, then K_{u_0} forms a solid cone in E_{u_0} which is normal. Moreover, the norm is monotone in E_{u_0}: it follows from $\theta \leqslant x \leqslant y$ that $\|x\|_{u_0} \leqslant \|y\|_{u_0}$.

If the cone K does not have the property of normality, then E_{u_0} cannot have the property of completeness. We denote the completion of E_{u_0} with respect to the u_0-norm by $\bar{E}_{u_0}$ and the closure of K_{u_0} in $\bar{E}_{u_0}$ by $\bar{K}_{u_0}$.

Theorem (1.4). *$\bar{K}_{u_0}$ is a solid and normal cone in $\bar{E}_{u_0}$; the norm is monotone in $\bar{E}_{u_0}$.*

1.4. *Linear positive functionals*

The linear functional $f(x)$ is called *positive* if $f(x) \geqslant 0$ for $x \in K$. All the linear positive functionals form the cone K^* in the conjugate space E^* (if K is reproducing). The set of positive functionals is sufficiently "extensive": for every $x_0 \in K (x_0 \neq \theta)$ there can be found a positive functional $f(x)$, such that $f(x_0) > 0$; for any two elements $x_0, y_0 \in K$, there can be found a positive functional $f(x)$ such that $f(x_0) \neq f(y_0)$. If the space E is separable, then a positive linear functional $f(x)$ can be found such that $f(y) > 0$ for $x \in K$ and $x \neq \theta$.

The functional $f(x)$ is called *uniformly positive* if $f(x) \geqslant a\|x\|$ $(x \in K)$. The cone K, by definition, *allows plastering* if there exists a cone K_1 such that every non-zero element $x_0 \in K$ is an interior point of the cone K_1 and, moreover, lies in the cone K_1 together with a spherical neighborhood of ra-

dius $b\|x_0\|$ where b does not depend on x_0. Every cone allowing plastering is normal but not every normal cone allows plastering.

Theorem (1.5). *The cone K allows plastering if and only if there is a uniformly positive linear functional defined on K.*

The cone of non-negative functions in the spaces C and L_p $(1<p<\infty)$ does not allow plastering. The cone of non-negative functions allows plastering in the space L.

Let F be a closed convex bounded set which does not contain the zero θ of the space E. We denote by $K(F)$ the collection of all elements of the form $\alpha x(\alpha \geqslant 0,\ x \in F)$. The sets $K(F)$ are cones allowing plastering.

1.5. *Regular cones*

The cone K is called *regular* if every non-decreasing sequence $x_n(x_1 \leqslant x_2 \leqslant \ldots x_n \leqslant \ldots)$, bounded above by some element z $(x_n \leqslant z,\ n=1, 2, \ldots)$, converges with respect to the norm.

Theorem (1.6). *Every regular cone is normal.*

The regularity of a cone does not follow from its normality.

The cone K is called *fully regular* if every non-decreasing sequence x_n bounded with respect to the norm $(\|x_n\| \leqslant M,\ n=1, 2, \ldots)$ converges with respect to the norm.

Theorem (1.7). *Every fully regular cone is regular.*

The cone of non-negative functions is completely regular in the spaces L_p $(1 \leqslant p<\infty)$; the cone of non-negative functions does not even have the property of regularity in the space C. Regular cones but not fully regular cones exist in C. The regular solid cone is fully regular (Theorem 1.8).

Theorem (1.9). *If the space E is weakly complete, then every regular cone is fully regular in this space.*

1.6. *Criteria for regularity of a cone*

The positive $(f(x) \geqslant 0$ for $x \in K)$ but not necessarily linear functional $f(x)$ is called *strictly increasing* if for arbitrary $h_n \in K$ $(n=1, 2, \ldots)$ it follows from $\|h_n\| \geqslant \varepsilon_0 > 0$ that $f(h_1 + \ldots + h_n) \to \infty$.

Theorem (1.10). *If a strictly increasing functional can be defined on K which is bounded on every sphere, then the cone K is fully regular.*

Theorem (1.11). *If a strictly increasing functional $f(x)$ can be defined on the cone K which is monotone (it follows from $\theta \leqslant x \leqslant y$ that $f(x) \leqslant f(y)$), then the cone K is regular.*

Theorem (1.12). *If the cone K allows plastering, then it is fully regular.*

The converse statement is not true since the cone of non-negative functions is fully regular in the spaces $L_p (1 < p < \infty)$ but does not allow plastering.

1.7. *Minihedral cones*

The set $\mathfrak{M} \subset E$ is called *bounded* if there exists an element z such that $x \leqslant z (x \in \mathfrak{M})$. The element z is called an *upper bound* for $\mathfrak{M}$. If in the set $\mathfrak{N}$ of all upper bounds there is an element z_0 such that $z_0 \leqslant z$ for all $z \in \mathfrak{N}$, then z_0 is called the *least upper bound* of the set $\mathfrak{M}$.

The cone K is called *minihedral* if every finite number of elements $x_1, \ldots, x_n \in E$ has a least upper bound $z_0 = \sup\{x_1, \ldots, x_n\}$.

Theorem (1.14). *If the cone K is minihedral and regular, then every bounded sequence has a least upper bound.*

The cone K is called *strongly minihedral* if every bounded set has a least upper bound.

Theorem (1.15). *Let there be defined on a minihedral and regular cone K some strictly monotone functional $f(x)$ (it follows from $x \leqslant y$ and $x \neq y$ that $f(x) < f(y)$) having the property that for every non-decreasing sequence x_n it follows from $\|x_n - x^*\| \to 0$ that $f(x_n) \to f(x^*)$.*
Then the cone K is strongly minihedral.

The cone K of non-negative functions is strongly minihedral in the spaces L_p $(1 \leqslant p < \infty)$. The cone of non-negative functions is minihedral in the space C but does not possess the property of strong minihedralness (moreover, not every monotone and bounded sequence has a least upper bound in C).

1.8. *Spaces with two cones*

Let two cones K_0 and K be given in E where $K_0 \subset K$. We denote by $K_0\langle x_0, y_0\rangle$ the collection of elements $x \in K_0$, satisfying the inequalities $x_0 \leqslant x \leqslant y_0$ (the partial ordering is introduced in E by means of the "larger" cone K).

Theorem (1.16). *A necessary and sufficient condition for the sets* $K_0\langle x_0, y_0\rangle$ *to be bounded for arbitrary* $x_0, y_0 \in K$ *is that the norm be semi-monotone on* K_0.

In a natural way the concept of *K-regularity* of the cone K_0 (monotone and bounded sequences of elements of K_0 converge with respect to the norm) and *full K-regularity* of the cone K_0 (monotone and bounded sequences of K_0 converge with respect to the norm) is introduced in spaces with two cones. Natural connections occur between the properties of K-regularity and full K-regularity and usual regularity and full regularity. Analogies of the theorems cited in section 1.5 occur.

Chapter 2. Linear positive operators

2.1. *Linear u_0-positive operators*

The linear operator A, acting in the Banach space E with a cone K, is called *positive* if it leaves the cone K invariant, that is, if it follows from $x \geqslant \theta$ that $Ax \geqslant \theta$. Every positive linear operator is *monotone* in the sense that $x \leqslant y (x, y \in E)$ implies the inequality $Ax \leqslant Ay$.

The positive linear operator A is called *strongly positive* if the cone K is solid and if for every non-zero $x \in K$ there can be found a natural number p such that $A^p x$ is an interior element of the cone K.

The linear integral operator

$$Ax(t) = \int_{\Omega} k(t, s) x(s) ds , \tag{2.1}$$

acting in a functional space E (for example, C or L_p), which is partially ordered by means of the cone K of non-negative functions on Ω will be positive if the kernel $k(t, s)$ is non-negative. If the kernel $k(t, s)$ is positive and continuous, then A is strongly positive in the space C.

Let u_0 be a fixed non-zero element of K. The positive linear operator A is called *u_0-bounded above* if for every $x \in K$ there can be found a natural number p such that $A^p x \leqslant \beta u_0 (\beta > 0)$. If the cone K is solid and u_0 is an interior element of K, then every linear positive operator is u_0-bounded above.

The operator A is called *u_0-bounded below* if for every non-zero $x \in K$ there can be found a natural number p such that $A^p x \geqslant \alpha u_0 (\alpha > 0)$. If for every $x \in K (x \neq \theta)$ a number n can be found such that $\alpha u_0 \leqslant A^n x \leqslant \beta u_0 (\alpha, \beta > 0)$,

then the linear operator A is called *u_0-positive*. Every strongly positive operator is an example of a u_0-positive operator.

Theorem (2.2). *If the operator A is simultaneously u_0-bounded above and below, then it is u_0-positive.*

The collection of elements $x \in K$ such that $au_0 \leqslant x \leqslant \varrho a u_0$ for some $a \geqslant 0$ is denoted by $K_{u_0,\varrho}$. $K_{u_0,\varrho}$ is a cone which allows plastering if K is normal (Theorem 1.13).

Theorem (2.1). *Let the linear operator A transform the cone K into the cone $K_{u_0,\varrho}$ and let it follow from $Ax=\theta (x \in K)$ that $x=\theta$. Then the operator A is u_0-positive on every cone $K_{u_0,\varrho+\varepsilon}$.*

The linear operator, leaving the cone K invariant, is called *uniformly positive* if $\|Ax\| \geqslant b\|x\| \, (x \in K)$ where $b>0$. The unit operator I can serve as an example of a uniformly positive operator.

Theorem (2.4). *If a uniformly positive and completely continuous operator can be defined on the cone K, then the cone K allows plastering.*

For some cases, we can deduce the continuity of a linear operator from its positiveness.

Theorem (2.3). *Let E_1 and E_2 be two Banach spaces with the cones K_1 and K_2, respectively, the first of which is reproducing and the second, normal. Let the additive and homogeneous operator A, acting from E_1 to E_2, be positive in the sense that $AK_1 \subset K_2$. Then A is continuous.*

2.2. *Existence of a characteristic vector*

The non-zero element x is called a *characteristic vector* of the operator A if $Ax=\lambda x$ where λ is some real number which is called the *characteristic value* of the operator A. If the characteristic vector belongs to K then it is called a *positive* characteristic vector; the corresponding characteristic value λ is called a *positive* characteristic value. Every positive linear operator, acting in a finite-dimensional space, has a positive characteristic vector. In the case of infinite-dimensional spaces this fact does not hold. For example, the linear integral operator

$$Ax(t)=\int_0^t k(t, s)x(s)\,ds \tag{2.2}$$

with a continuous positive kernel $k(t, s)$ does not have only positive but also other characteristic vectors in the spaces C and L_p in which this operator is not only positive but also completely continuous.
In connection with the example indicated the problem arises about the additional conditions for which a positive linear operator has characteristic vectors.

Theorem (2.5). *Let the linear positive operator A be completely continuous. Let the relation*

$$A^p u \geqslant \alpha u \qquad (\alpha > 0) \tag{2.3}$$

be satisfied for some non-zero element $u(-u \bar{\in} K;\ u = v - w,\ v \in K,\ w \in K)$ where p is some natural number. Then the operator A has at least one positive characteristic vector where the corresponding positive characteristic value is greater than or equal to $\sqrt[p]{\alpha}$.
Example (2.2) shows that condition (2.3) is essential.

Theorem (2.6). *Every uniformly positive completely continuous operator has at least one positive characteristic vector.*

In conclusion we consider the case when the space E is weakly complete and the unit sphere is compact in E. If in this case the cone K allows plastering, then (Theorem 2.7) there is a positive characteristic vector for every linear positive operator. Example (2.2) shows that each of the enumerated conditions is essential: the operator (2.2) is positive in the spaces $L_p (1 < p < \infty)$ which are weakly complete and weakly compact, but the operator (2.2) does not have characteristic vectors in L_p and the situation is that the cone of non-negative functions does not allow plastering in $L_p (1 < p < \infty)$; the operator (2.2) is positive in L in which the cone of non-negative functions allows plastering, but there are no characteristic vectors in L and the space L does not have the property of weak completeness and weak compactness.
A number of more particular criteria exist for the existence of a characteristic vector in the cone (Theorems 2.8 and 2.9).

2.3. *Simplicity of a positive characteristic value*
All the propositions of the present and following sections carry over to the case when the cone K is reproducing. In all theorems it is assumed that the linear positive operator A being considered has a positive characteristic value.

The characteristic value λ_0 of a linear operator A is called *simple* if all the solutions of the equations $(A-\lambda_0 I)^n x=\theta$ $(n=1, 2, \ldots)$ are simultaneously solutions of the equation $Ax=\lambda_0 x$ where the set of solutions of the last equation is one-dimensional.

Theorem (2.10 and 2.11). *Let the operator A be u_0-positive. Then*
(a) *the positive characteristic value λ_0 of the operator A is simple*;
(b) *the operator A has a unique (to within the norm) characteristic vector in the cone.*
The subspace $\Pi \subset E$ is called an *invariant subspace* of the operator A if $A\Pi \subset \Pi$.

Theorem (2.12). *Let the operator A be u_0-positive and let Π be an invariant subspace of the operator A to which there does not belong a positive characteristic vector of the operator A. Let one of the following three conditions be satisfied*:
(a) *the subspace Π is finite-dimensional,*
(b) *the operator A is completely continuous,*
(c) *the subspace Π is weakly complete, the unit sphere in Π is weakly compact, the cone $K \cap \Pi$ allows plastering.*
Then the intersection of the subspace Π with the cone K consists of the single point θ.

Both theorems formulated above carry over, for example, to integral operators if some iterate $k_N(t, s)$ of the kernel $k(t, s)$ satisfies the inequalities

$$u_0(t) \leqslant k_N(t, s) \leqslant \varrho u_0(t) \qquad (t, s \in \Omega);$$

the last inequalities guarantee the u_0-positiveness of the operator.

2.4. *Comparison with other characteristic values*
The complex number $\lambda=\sigma+i\tau$ is called a *characteristic value* of the linear operator A, acting in a real Banach space E, if the formal equality

$$Ax+iAy=(\sigma+i\tau)(x+iy)$$

holds, that is, the two equalities

$$Ax=\sigma x-\tau y, \quad Ay=\sigma y+\tau x$$

hold.

Theorem (2.13). *Let λ_0 be a positive characteristic value of the u_0-positive*

operator A. Then every other characteristic value of the operator A satisfies the inequality $|\lambda| < \lambda_0$.

For the u_0-bounded above (Theorem 2.14) and u_0-bounded below (Theorem 2.15) operators weaker statements are valid.

2.5. *Non-homogeneous linear equations*
We consider the equation

$$\lambda x = Ax + f \tag{2.4}$$

with the positive operator A. Let λ_0 be the least upper bound of all the characteristic values of the operator A.

Theorem (2.16). *If $\lambda > \lambda_0$ and $f \in K$ then equation* (2.4) *has a unique solution in K.*
If $\lambda \leqslant \lambda_0$ and $A^n f \geqslant \varepsilon u_0 (\varepsilon > 0)$ where u_0 is a non-zero element of K such that $A^p u_0 \geqslant \lambda_0^p u_0$, then equation (2.4) *does not have solutions in K.*

The second part of the statement of this theorem can be considered as a proposition about the impossibility of some inequalities.
In the sequel we shall write $x \overline{\leqslant} y$ if $y - x \overline{\in} K$.

Theorem (2.17, 2.18 and 2.19). *Let the operator Au_0 be bounded below where $Au_0 \geqslant \lambda_0 u_0$. Then for an arbitrary non-zero element $x \in K$ the relation $Ax \overline{\leqslant} \lambda x$ is satisfied for $\lambda < \lambda_0$ and it follows from $Ax \leqslant \lambda_0 x (x \in K)$ that $x = ku_0$ and $Au_0 = \lambda_0 u_0$.*
Let Au_0 be bounded above and $Au_0 \leqslant \lambda_0 u_0$. Then for an arbitrary element $x \in K$ the relation $Ax \overline{\geqslant} \lambda x$ is satisfied for $\lambda > \lambda_0$.
Finally, let A be u_0-positive and $Au_0 = \lambda_0 u_0$. Then for an arbitrary non-zero element $x \in K$ which is not collinear with u_0, the elements $\lambda_0 x$ and Ax are incomparable:

$$Ax \overline{\leqslant} \lambda_0 x, \quad Ax \overline{\geqslant} \lambda_0 x.$$

Chapter 3. Differentiability with respect to a cone

3.1. *Derivatives with respect to a cone*
In the study of non-linear operators an essential role is played by the concepts of the derivatives which are defined in the usual way: by the choice of the principal linear parts of the increments. In connection with

this for infinite-dimensional spaces various concepts of the limit exist (strong and weak) and different concepts of the derivatives must be considered. We give basic definitions.

Let

$$A(x_0+h)-Ax_0=A'(x_0)h+\omega(x_0, h)\,, \tag{3.1}$$

where $A'(x_0)$ is a linear operator. The operator $A'(x_0)$ is called a *weak Gâteaux derivative* of the operator A at the point x_0 if for every linear functional $f(x)$ and every fixed $h_0\in E$ the equality $\lim_{t\to 0}(1/t)f[\omega(x_0, th_0)]=0$ is satisfied. The operator $A'(x_0)$ is called a *strong Gâteaux derivative* if

$$\lim_{t\to 0}\frac{1}{t}\,\|\omega(x_0, th_0)\|=0$$

for an arbitrary fixed $h_0\in E$. The operator $A'(x_0)$ is called a *strong* or *weak Fréchet derivative* if $(1/\|h\|)\,\omega(x_0, h)$ approaches zero with respect to the norm strongly or weakly, respectively, as $\|h\|\to 0$.

If only the elements $h\in K$ are considered in the given definitions, then instead of a Gâteaux derivative we obtain the *definitions of derivatives with respect to the cone K* (*weak or strong*) and instead of derivatives according to Fréchet we obtain *Fréchet derivatives with respect to a cone* (*weak or strong*).

The assumption about the existence of derivatives with respect to a cone is essentially a lesser restriction than the assumption about the existence of corresponding Gâteaux or Fréchet derivatives. However, for some cases differentiability according to Fréchet follows from the existence of derivatives with respect to a cone.

The cone K, by definition, is *non-oblate* if a number $M>0$ can be found such that for every $x\in E$ an element $u=u(x)\in K$ can be found such that $x\leqslant u$ and $\|u\|\leqslant M\|x\|$. The cones of non-negative functions are non-oblate in the spaces C and L_p. The family of non-oblate cones coincides with the family of reproducing cones.

Theorem (3.1). *Let the cone K be non-oblate. Let the operator A have a weak derivative $A'(x)$ with respect to the cone K where $A'(x)$ continuously (with respect to a norm of linear operators) depends on x. Then $A'(x)$ is a strong Fréchet derivative.*

Theorem (3.2). *A strong Fréchet derivative with respect to a non-oblate cone K of a completely continuous operator is a completely continuous operator.*

3.2. *Derivative at infinity*

Let

$$Ax = A'(\infty)x + \omega(x), \tag{3.2}$$

where $A'(\infty)$ is some linear operator. The operator $A'(\infty)$ is called a *weak derivative at infinity* for the operator A if the relation $(1/t)f[\omega(tx_0)]$ approaches zero as $t \to \infty$ for every fixed non-zero $x_0 \in E$ and an arbitrary linear functional $f(x)$. The operator $A'(\infty)$ is called a *strong derivative at infinity* if the ratio $(1/t)\|\omega(tx_0)\|$ approaches zero as $t \to \infty$ for every non-zero $x_0 \in E$. The operator A is called *asymptotically linear* (*strongly* or *weakly*) if the ratio $(1/\|x\|)\omega(x)$ approaches zero as $\|x\| \to \infty$ (strongly or weakly); the operator $A'(\infty)$ in this case is called an *asymptotic derivative* (*weak* or *strong*).

If only the elements $x \in K$ are considered in the given definitions, then we arrive at the definitions of *derivatives* and *asymptotic derivatives with respect to the cone K.*

Under natural assumptions the derivative $A'(\infty)$ on infinity can be obtained as the limit of the usual derivatives $A'(x)$ as $\|x\| \to \infty$. Thus, for example, we arrive at

Theorem (3.3). *The operator* $\boldsymbol{B}$ *is a strong asymptotic derivative of the operator* A *if* A *has a weak Gâteaux derivative* $A'(x)$ *at every point* x *with a sufficiently large norm and if* $\|A'(x) - \boldsymbol{B}\| \to 0$ *as* $\|x\| \to \infty$.

Theorem (3.4). *The operator* $\boldsymbol{B}$ *is a strong asymptotic derivative with respect to the cone* K *if* $\boldsymbol{B}$ *is the limit, with respect to the norm of operators as* $\|x\| \to \infty$ *and* $x \in K$, *of weak Gâteaux derivatives* $A'(x)$ *with respect to a cone and if the inequality*

$$\|Ax\| \leqslant a(R) \qquad (x \in K,\ \|x\| \leqslant R)$$

is satisfied for every R.

The theorem analogous to the second theorem of the preceding subsection is valid for asymptotic derivatives with respect to a cone of completely continuous operators.

3.3. *Inequalities for elements with small norms*

It is assumed here that $A\theta = \theta$. The question is considered about whether there exist non-zero elements x_0 with small norms such that $Ax_0 \geqslant x_0$ or $Ax_0 \leqslant x_0$. If $Ax_0 \geqslant x_0$, then we shall say that the element x_0 is "progressing"

under the application of the operator A. If $Ax_0 \leqslant x_0$, then the element x_0 is "retrogressing". If $Ax \overline{\leqslant} x$ for all non-linear elements x of K with a sufficiently small norm, then we shall call the point θ *repelling* for the operator A. If the relation $Ax \overline{\geqslant} x$ is satisfied for non-zero elements $x \in K$ with a small norm, then we call the point θ *attracting*.

Since the values of the operator A are studied on elements with a small norm, then it is natural to expect that for basic cases the answer to the formulated question can be given in terms involving the properties of the derivative $A'(\theta)$.

We assume that $A'(\theta)$ is a strong derivative of the operator A with respect to a cone, and let

$$A'(\theta)h_0 = \lambda_0 h_0 \qquad (h_0 \in K,\ h_0 \neq \theta,\ \lambda_0 > 0)\,. \tag{3.3}$$

Theorem (3.4 and 3.6). *Let one of the following two conditions be satisfied*:
(a) *The cone K is solid, h_0 is an interior element of the cone K*;
(b) $\|A(\varepsilon h_0) - \varepsilon A'(\theta)h_0\|_{h_0} = o(\varepsilon)$.
Then the elements εh_0 are progressing for small $\varepsilon > 0$ if $\lambda_0 > 1$ and retrogressing if $\lambda_0 < 1$.

We further assume that A allows the representation

$$Ax = A'(\theta)x + Cx + Dx\,, \tag{3.4}$$

where C is a homogeneous operator of degree $k > 1 (C(tx) \equiv t^k Cx)$, and D is an operator of a higher order of smallness ($\|D(tx)\| = o(t^k)$).

Theorem (3.5 and 3.6). *Let $\lambda_0 = 1$. Let either condition (a) of the preceding theorem or condition (b) and the relation $\|D(\varepsilon h_0)\|_{h_0} = o(\varepsilon^k)$ be satisfied. Then the elements εh_0 are progressing for small $\varepsilon > 0$ if $\alpha h_0 \leqslant Ch_0 \leqslant \beta h_0$ $(\alpha, \beta > 0)$ and retrogressing if $\alpha h_0 \leqslant -Ch_0 \leqslant \beta h_0$ $(\alpha, \beta > 0)$.*

The operator A is called *uniformly h_0-bounded above* if there can be found a number $r = r(\varepsilon) > 0$ for every sufficiently small $\varepsilon > 0$ such that $Ax \leqslant \varepsilon h_0$ $(x \in K, \|x\| \leqslant r)$. If the cone K is solid and h_0 is an interior element K, then all the continuous operators satisfying the condition $A\theta = \theta$ at the point θ will be uniformly h_0-bounded above.

Theorem (3.7). *Let $A'(\theta)$ be a strong Fréchet derivative of the continuous operator A with respect to a cone which is monotone and uniformly h_0-bounded (h_0 satisfies the inequality* (3.3)). *Let one of the conditions* (a) *or* (b) *of the first theorem of the present section be satisfied where it is assumed that λ_0 is the greatest characteristic value of the operator $A'(\theta)$ for condi-*

tion (b). *Let the cone K be normal and $\lambda_0 < 1$.*
Then θ is an attracting point for the operator A.

This statement remains valid if the condition of complete continuity of the operator A is replaced by the assumption of weak continuity, and the condition of the normality of the cone is replaced by the assumption of weak completeness, and weak compactness of the sphere in E and the fact that K allows plastering.
The linear operator B, by definition, is *K-decomposable* if it has a characteristic vector in K and has an invariant subspace $E_1 \subset E$ of defect 1 which in common with K has only a zero element θ. This means that every element $x \in E$ is uniquely representable in the form

$$x = f(x)h_0 + x_1 \qquad (x_1 \in E_1), \tag{3.5}$$

where the linear functional $f(x)$ satisfies the conditions $f(h_0) = 1$ and $f(x_1) = 0$ for $x_1 \in E_1$. In this connection

$$Bx = \lambda_0 f(x) h_0 + Bx_1 .$$

Every u_0-positive linear completely continuous operator can serve as an example of a K-decomposable operator.

Theorem (3.8). *Let the strong Fréchet derivative $A'(\theta)$ be a K-decomposable operator with respect to a cone where the functional $f(x)$ is uniformly positive in the representation* (3.5). *Let $\lambda_0 < 1$.*
Then θ is an attracting point for the operator A.

Analogues of the last two theorems hold in which criteria are given such that θ is a repelling point for the operator A. The basic change in these analogies (Theorems 3.9 and 3.10) is contained in the transition from the inequality $\lambda_0 < 1$ to the inequality $\lambda_0 > 1$.

3.4. *Inequalities for elements with large norms*
The derivative $A'(\theta)$ can be used for a general characterization of the values of the operator A on elements x having a large norm. Statements occur (Theorems 3.11–3.13) which are analogous to the extensions given in the preceding section. The basic result can be formulated in the following manner.
Let the positive operator $A'(\infty)$ have a unique characteristic vector g_0:

$$A'(\infty) g_0 = \lambda_\infty g_0$$

in the cone K. Then the elements tg_0, generally speaking, are progressing provided $\lambda_\infty > 1$ and retrogressing if $\lambda_\infty < 1$. All the elements $x \in K$ with a large norm satisfy, generally speaking, the relation $Ax \lesseqgtr x$ for $\lambda_\infty > 1$ and the relation $Ax \gtreqless x$ for $\lambda_\infty < 1$.

Chapter 4. Existence of positive solutions

4.1. *Equations with monotone operators*

We denote by $\langle x_0, y_0 \rangle$ the collection of elements $x \in E$ such that $x_0 \leqslant x \leqslant u_0$; this collection is called a *conical segment.*

Theorem (4.1). *It suffices for the existence of at least one fixed point for a monotone operator A on $\langle x_0, u_0 \rangle$ that it transform $\langle x_0, u_0 \rangle$ into itself and that one of the following conditions be satisfied*:
(a) *The cone K is strongly minihedral*;
(b) *The cone K is regular, the operator A is continuous*;
(c) *The cone K is normal, the operator A is completely continuous*;
(d) *The cone K is normal, the space E is weakly complete, and the unit sphere is weakly compact in E, the operator A is weakly continuous.*

For the conditions of this theorem, as the examples show, the solution cannot be unique.
For condition (a) the fixed point x^* of the operator A is considered as a *supremum* of elements $x \in \langle x_0, u_0 \rangle$ such that $Ax \geqslant x$. For the remaining conditions the point x^* can be obtained as the limit of the successive approximations $x_n = Ax_{n-1}$ $(n = 1, 2, \ldots)$.
The applications of the theorem require the construction of a conical segment $\langle x_0, u_0 \rangle$ which is transformed by the operator A into itself. In other words, it is necessary to find elements x_0 and $u_0 (x_0 \leqslant u_0)$ such that $x_0 \leqslant Ax_0$ and $u_0 \geqslant Au_0$. If elements x_0 are found with a small norm and u_0 with a large norm, then the theorems of chapter 3 can be applied.
There exists a number of additional conditions which are different from conditions (a)–(d) for the existence of fixed points for monotone operators leaving a conical segment or another part of the cone invariant. We give one more statement.

Theorem (4.5). *Let the positive monotone and continuous operator A have a strong asymptotic derivative $A'(\infty)$ with respect to the cone K. Let the*

spectrum of the linear operator $A'(\infty)$ lie in a circle of radius $\varrho_0 < 1$. Let the cone K be fully regular.
Then the operator A has at least one fixed point on K.

4.2. *Equations with semi-monotone operators*
A more convenient way to prove the existence of fixed points for a positive operator A consists in the determination of convex sets $T \subset K$ which are invariant for A. If such sets exist and if we can succeed in finding them, then for a majority of cases it remains to apply Schauder's or Tikhonov's principle. The construction of invariant sets sometimes requires special constructions, but for basic cases this does not require the overcoming of any difficulties.

Theorem (4.7). *Let the positive completely continuous operator A have a strong asymptotic derivative $A'(\infty)$ with respect to a cone and let the spectrum of the linear operator $A'(\infty)$ lie in a circle $|\lambda| \leqslant \varrho_0$ where $\varrho_0 < 1$.*
Then the operator A has at least one fixed point in the cone K.

Theorem (4.9). *Let the positive completely continuous operator A satisfy the condition: there exists a number R such that*

$$Ax \geqq (1+\varepsilon)x \qquad (x \in K,\ \|x\| = R) \tag{4.1}$$

for all $\varepsilon > 0$.
Then the operator A has at least one fixed point in the cone K.

It is convenient to consider various majorants A^+ of the operator A for the verification of condition (4.1): if this condition is satisfied for the majorant, then it is satisfied for the operator A. We can carry out the verification of condition (4.1) for the operator A or its majorant A^+ basing it on the theorems developed in chapter 3.
We now assume that $A\theta = \theta$ and raise the question about the existence of a non-zero positive solution for the positive operator A. Sufficient conditions for the existence of such solutions found in this chapter follow from the following simple geometrical considerations.
Let the non-zero elements of the cone K with small norms be progressing under the application of the operator A and the elements with large norms be retrogressing. Let the cone K not only be normal but, moreover, let the norm be monotone. Then the operator A leaves the intersection of the cone K with some spherical shell $0 < r \leqslant \|x\| \leqslant R < \infty$ invariant. If A is completely continuous, then the existence of a fixed point follows from

Schauder's principle. The enumerated conditions are extraordinarily restrictive; however, we can succeed in replacing them by conditions which are simpler and easily verified.

Theorem (4.11). *Let the positive completely continuous operator $A(A\theta=\theta)$ have a strong Fréchet derivative $A'(\theta)$ and a strong asymptotic derivative $A'(\infty)$ with respect to a cone. Let the spectrum of the operator $A'(\infty)$ lie in a circle $|\lambda|\leqslant\varrho_0$ where $\varrho_0<1$. Let the operator $A'(\infty)$ have a characteristic vector in K to which there corresponds a characteristic value $\lambda_0>1$ and let $A'(\theta)$ not have characteristic vectors in K to which there corresponds a characteristic value equal to* 1.
Then A has at least one non-zero fixed point on K.

We shall say that the operator $A(A\theta=\theta)$ is a *compression* of the cone if condition (4.1) is satisfied and if a number $r>0$ can be found such that

$$Ax \overline{\leqslant} x \qquad (x\in K,\ \|x\|\leqslant r,\ x\neq\theta)\,. \tag{4.2}$$

Theorem (4.12). *Let the positive completely continuous operator A be a compression of the cone. Then A has at least one non-zero fixed point on K.*

The verification of condition (4.2) is simplified in a number of cases if minorants A^- of the operator A are taken into consideration.

4.3. *Auxiliary statements*

The last two theorems have a particularly simple meaning for the case when K is a one-dimensional cone, that is, when $Ax=f(x)$ is a non-negative function with a numerical variable. The condition for the existence of a positive fixed point in these theorems consists in the fact that the inequality $f(x)>x$ is satisfied for small x and the inequality $f(x)<x$ is satisfied for large x.

It is obvious for the one-dimensional case that it suffices for the existence of a fixed point that the inequality $f(x)<x$ be satisfied for small x and the opposite inequality $f(x)>x$ for large x. For these conditions there is a fixed point but, generally speaking, there are no invariant segments. Therefore, it is clear that the generalization to the infinite-dimensional case cannot be based on the direct application of the principles of a fixed point of the type coming under the heading of Schauder's or Tikhonov's principle if this generalization can be found and, thus, new topological theorems are necessary.

We give a basic statement on which the study of completely continuous operators is based.

Theorem (Lemma 4.5). *Let the completely continuous operator A be positive on the set $T=T\{x:x\in K,\ r\leqslant\|x\|\leqslant R\}$ and satisfy the conditions*

$$Ax\equiv v_0 \quad \text{for} \quad x\in K,\ \|x\|=r,$$

$$Ax\equiv u_0 \quad \text{for} \quad x\in K,\ \|x\|=R,$$

where $\|v_0\|<r<R<\|u_0\|$.
Then A has at least one fixed point on T.

4.4. *Fixed points of operators which expand the cone*
The positive operator $A(A\theta=\theta)$ is called an *expansion* of the cone K if numbers r and R can be found such that

$$Ax\geqq(1+\varepsilon)x \qquad (x\in K,\ \|x\|\leqslant r) \tag{4.3}$$

and

$$Ax\leqq x \qquad (x\in K,\ \|x\|\geqslant R) \tag{4.4}$$

for all $\varepsilon>0$.

Theorem (4.14). *Let the positive completely continuous operator A be an expansion of the cone. Then A has at least one non-zero fixed point on K.*

The verification of conditions (4.3) and (4.4) can be simplified if the majorants A^+ and minorants A^- of the operator A are taken into consideration.

Theorem (4.16). *Let the strong Fréchet derivative $A'(\theta)$ not have positive characteristic values greater or equal to* 1 *with respect to a cone. Let the strong asymptotic derivative $A'(\infty)$ have a positive characteristic value $\lambda_\infty>1$ and not have* 1 *as its positive characteristic value.*
Then the operator A has at least one fixed non-zero point on K.
We further assume that the space E is weakly complete, the unit sphere is weakly compact in E and the cone K allows plastering. With these assumptions all the statements of this chapter remain valid provided the assumption about the complete continuity of the operator A is replaced by the assumption of its weak continuity.

Chapter 5. Continuous branches of positive solutions

5.1. *Positive solutions of equations with a parameter*

The equation

$$x = A(x; \mu) \tag{5.1}$$

is studied with the positive operator $A(x; \mu)$ depending on the numerical parameter μ. It is assumed that $A(x; \mu)$ is completely continuous as an operator in two variables.

The results of the preceding chapter are applied for the proof of the existence of solutions for fixed values of the parameter μ. We give one statement as an example.

We assume that the operator A has a strong Fréchet derivative $A'(\theta; \mu)$ with respect to a cone for every fixed μ of the form

$$A'(\theta; \mu) = \mu A'(\theta)$$

and a strong asymptotic derivative $A'(\infty; \mu)$ of the form

$$A'(\infty; \mu) = \mu A'(\infty) .$$

Theorem (5.1). *Let each of the linear operators $A'(\theta)$ and $A'(\infty)$ have a unique characteristic value which we denote by λ_0 and λ_∞ respectively. Let $A(\theta; \mu) \equiv \theta$.*

Then equation (5.1) *has at least one non-zero solution $x(\mu)$ in the cone K for every $\mu \in (1/\lambda_0, 1/\lambda_\infty)$.*

If the solution $x(\mu)$ is unique, then it depends continuously on the parameter μ. In the general case, the solution is not $x(\mu)$ unique; therefore, it is necessary to understand continuous dependence in some general sense.

The set $\mathfrak{N} \subset E$ forms, by definition, a *continuous branch connecting a bounded closed set $F \subset E$ with the set F_1* if the intersection of the set $\mathfrak{N}$ with the boundary of every bounded domain containing F and not having common points with F_1 is not empty.

We denote by $X(\mu)$ the set of all solutions $x(\mu) \in K$ of the equation (5.1) for a given fixed value of the parameter μ. If $A(x; \mu)$ is completely continuous, then every $X(\mu)$ is closed and locally compact. It turns out (Theorem 5.4) that under the above mentioned conditions (with additional non-essential assumptions) *the set $\mathfrak{N}$ of all solutions of the equation* (5.1) *forms a continuous branch connecting the sets $X(\mu_1)$ and $X(\mu_2)$ for all μ of the interval*

$$[\mu_1, \mu_2] \subset \left(\frac{1}{\lambda_0}, \frac{1}{\lambda_\infty}\right) .$$

For a special case, when the equation (5.1) has the form

$$x=\mu Ax\,, \tag{5.2}$$

the non-zero solutions $x(\mu)$ are called *characteristic vectors* of the non-linear operator A. This term assumes by analogy with the theory of linear operators the role of characteristic vectors in the theory of non-linear equations which, however, is highly modest because of the absence of the principle of superposition. Characteristic vectors or characteristic functions are only non-zero solutions.
Equation (5.2) is often written in the form

$$Ax=\lambda x\,. \tag{5.3}$$

The values λ, for which the equation (5.3) has non-zero solutions, are called *characteristic values*. The collection of characteristic values are called the *spectrum* of the operator A.

Theorem (5.5). *Let Γ be the boundary of an open bounded set $\mathfrak{G}\subset E$ for which θ is an interior element. Let there be defined on $\Gamma\cap K$ a positive completely continuous operator A where*

$$\inf_{x\in\Gamma\cap K}\|Ax\|>0\,. \tag{5.4}$$

Then the operator A has at least one characteristic vector x_0, $Ax_0=\lambda_0 x_0$, on $\Gamma\cap K$ to which there corresponds the positive characteristic value λ_0.

The positive vectors of the operator A form, by definition, a *continuous branch $\mathfrak{N}$ of length r* if this collection $\mathfrak{N}$ of all positive characteristic vectors has a non-empty intersection with the boundary Γ of every set containing θ and contained in the sphere $\|x\|<r$. It follows from the last theorem that the positive characteristic vectors of a positive completely continuous operator A forms a continuous branch of some length r if $A\theta\neq\theta$.

5.2. *Some topological theorems*
The last two (mentioned in the preceding subsection) theorems are of a topological nature. It turns out that their proofs require either the application of the theory of power mappings or the application of the theory of the rotation of a vector field.

5.3. *Operators with monotone minorants*
Condition (5.4) is restrictive; therefore, theorems for the existence or

characteristic vectors exist for which this condition is not present are important. The basic method for proving such theorems consists in the construction of those operators A_n, which satisfy condition (5.4) and which converge to the operator A. Then there exists a sequence x_n such that $A_n x_n = \lambda_n x_n$, where the λ_n are some unknown numbers. The sequence $A_n x_n$ can be assumed to be convergent if A is completely continuous. Therefore, for the proof of the existence of a characteristic vector for the limit operator A it is necessary to be able to accomplish a passage to the limit with respect to a subsequence of indices and to this end it suffices that the numbers λ_n be bounded below by some positive number. The proof of the boundedness below of the numbers represents the basic difficulty. We give one basic result.

Theorem (5.8). *Let the positive completely continuous operator A have a monotone minorant B on the set $K_r = K_r\{x : x \in K, \|x\| \leqslant r\}$: B is monotone and*

$$Ax \geqslant Bx \qquad (x \in K_r)\,. \tag{5.5}$$

Let there exist a non-zero element $u \in K$ such that

$$B(tu) \geqslant \alpha tu \qquad (0 \leqslant t \leqslant \gamma)\,, \tag{5.6}$$

where $\alpha > 0$ and γ is a number such that $x \geqslant tu$ follows from $\|x\| \leqslant r$ and $t \leqslant \gamma$. Then A has a continuous branch of characteristic vectors of length r in the cone K.

If B is a linear operator, then condition (5.6) assumes the simple form: $Bu \geqslant \alpha u$.

The concept of a continuous branch of solutions finds an application in the questions which are connected with the extendibility of solutions. The general method for the extension of a solution with respect to a parameter is connected essentially with the local uniqueness of the solution for every value of the parameter being considered. Clearly, such uniqueness does not hold in non-linear problems: the solutions are branched and again merged for a change in the parameter, and appear as a continuum of solutions and so on. However, the concept of a continuous branch allows the description of a general principle of topological extension (or topological inclusion) covering the case when the uniqueness of the solutions cannot be guaranteed for a change of the parameter. We describe this general principle.

We assume that it is necessary to prove the existence of a solution for the equation $x = Ax$. We attempt to introduce in the equation the numerical

parameter μ in order for the new equation (5.1) to be transformed into the original equation for $\mu=1$ and to be able to establish the existence of a continuous branch $\mathfrak{N}$ of solutions which correspond to some unknown values of the parameter μ for the equation (5.1). If the parameter μ can be introduced, then we can prove in a number of cases that the values of the parameter μ for which equation (5.1) is solvable approach some limits μ_0 and μ_∞ when the norms of the solutions, for example, approach zero and infinity. We can deduce from this fact that the solutions form a continuous branch (this deduction can be made under natural restrictions) that the solutions exist for all $\mu\in(\mu_0, \mu_\infty)$. Thus, the existence of a solution is proved if $1\in(\mu_0, \mu_\infty)$.

The carrying out of the described plan, in particular, requires the determination of those limits to which the values of the parameter approach when the norms of the solutions approach zero or increase without limit. If the solutions are positive characteristic vectors, then the limit values of the parameter coincide, generally speaking, with positive characteristic values of the operators $A'(\theta)$ and $A'(\infty)$.

Chapter 6. Equations with concave operators

6.1. *Uniqueness of a positive solution and convergence of the successive approximations*

Let u_0 be some fixed non-zero element of the cone K.

We call the operator A *concave* if the inequalities

$$\alpha(x)u_0\leqslant Ax\leqslant\beta(x)u_0 \tag{6.1}$$

are valid for an arbitrary non-zero $x\in K$ where $\alpha(x)$ and $\beta(x)$ are positive and if the relations

$$A(tx)\geqslant tAx \qquad (0\leqslant t\leqslant 1) \tag{6.2}$$

are valid for every element $x\in K$ such that $\alpha_1(x)u_0\leqslant x\leqslant\beta_1(x)u_0$ ($\alpha_1(x)>0$, $\beta_1(x)>0$).

Theorem (6.1). *Let x_1 and x_2 be non-zero positive solutions of the equation*

$$Ax=\lambda x \tag{6.3}$$

with a concave and monotone operator A corresponding to distinct values λ_1 and λ_2 of the parameter λ. Then it follows from $\lambda_1<\lambda_2$ that $x_1\geqslant x_2$.

Under the conditions of this theorem the equation has non-zero solutions and one of the following additional conditions is satisfied for all $\lambda \in (\lambda_1, \lambda_2)$:
(a) The cone K is strongly minihedral;
(b) the cone K is regular and the operator A is continuous;
(c) the cone K is normal and the operator A is completely continuous.
The concave operator A is called *u_0-concave* if condition (6.2) is replaced by a more severe restriction: for every number $t_0 \in (0, 1)$ there can be found a number $\eta = \eta(x; t_0) > 0$ such that

$$A(t_0 x) \geqslant (1+\eta) t_0 A x . \tag{6.4}$$

Theorem (6.3). *Equation* (6.3) *with a monotone and u_0-concave operator A does not have two non-zero solutions for any value of the parameter λ in the cone K.*
The statement of this theorem is also valid for operators A, some power A^p of which is a u_0-concave operator. A family of operators is found whose square is a u_0-concave operator.
The criteria for concavity which are expressed in the properties of derivatives are valid for non-linear operators and are analogous to corresponding criteria for concavity of scalar functions.

Theorem (6.6). *Let x^* be a unique non-zero positive solution of the equation $Ax = x$ with a concave and monotone operator A. Let either one of the following conditions be satisfied*:
(a) *The cone K is regular and the operator A is continuous*;
(b) *The cone K is normal and the operator A is completely continuous.*
Then the successive approximations

$$x_n = A x_{n-1} \qquad (n = 1, 2, \ldots) \tag{6.5}$$

converge to x^* with respect to the norm no matter what the initial approximation $x_0 \in K$, $x_0 \neq \theta$ is.

If the operator A is monotone and u_0-concave and the cone K is normal, then the statement of this theorem remains valid without the assumption about the continuity of the operator A. Moreover, the successive approximations (6.5) converge to x^* with respect to the u_0-norm (Theorem 6.7) for an arbitrary $x_0 \in K (x_0 \neq \theta)$ for equations with u_0-concave operators.

6.2. *Existence of a continuum of characteristic vectors*
The existence of a continuous branch of solutions for some values λ for

the equations (6.3) with concave completely continuous operators follows from the general results of chapter 5. Together with the uniqueness theorems this reduces to the following result:

Theorem (6.8). *The positive spectrum of a monotone and u_0-concave completely continuous operator A forms the segment $(\lambda_\infty, \lambda_\theta)$. The solutions $x(\lambda)(\lambda_\infty < \lambda < \lambda_\theta)$ of the equation* (6.3) *form a single-valued vector function which is continuous and decreasing.*

The existence of positive characteristic vectors is not proved for concave operators which do not have the property of complete continuity. But if it is known that there is one characteristic vector, then for the case of fully regular cones all the statements of the formulated theorem are valid.

The bounds λ_θ and λ_∞ of the positive spectrum of the operator A are defined (Theorems 6.10 and 6.11), generally speaking, as a positive characteristic value of a strong Fréchet derivative $A'(\theta)$ with respect to a cone and a strong asymptotic derivative $A'(\infty)$ with respect to a cone.

All the statements carry over to equations with concave operators and are generalized in a natural way to the case when the operator A is defined not on the entire cone K but on some closed set $T \subset K$ which possesses the property that every element x belongs to T together with all the elements $tx (0 \leqslant t \leqslant 1)$.

6.3. *Equations with convex operators*

The operator A is called *u_0-convex* if condition (6.1) is satisfied and if a positive number $\eta = \eta(x, t_0)$ can be found such that

$$A(t_0 x) \geqslant (1-\eta)\, t_0 Ax \tag{6.6}$$

for every element $x \in K$ such that $\alpha_1(x)u_0 \leqslant x \leqslant \beta_1(x)u_0$ $(\alpha_1(x) > 0, \beta_1(x) > 0)$ and every for number $t_0 \in (0, 1)$.

For equations with convex operators many statements do not hold which are true for equations with concave operators. In particular, the theorem about the uniqueness of a positive solution is not true without essential additional assumptions.

Uniqueness principle. *Let x^* and x^{**} be non-zero positive solutions of the equation* (6.3) *with a monotone u_0-convex operator A. Let the cone K be solid and let one of the elements $x^* - x^{**}$, $x^{**} - x^*$ be either zero or an interior element of the cone. Then $x^* = x^{**}$.*

6.4. *Applications to the problem about points of bifurcation*
We assume that $A\theta=\theta$. Then the equation (6.3) has the trivial zero solution θ for all values of the numerical parameter λ. The number $\lambda_0(\lambda_0>0)$ is called a *point of bifurcation* for the operator A if to every number $\varepsilon>0$ there corresponds a value λ which satisfies the inequality $|\lambda-\lambda_0|<\varepsilon$ and for which the equation (6.3) has at least one non-zero solution $x(\lambda)$ satisfying the condition $\|x(\lambda)\|<\varepsilon$.
We assume that the operator $A(A\theta=\theta)$ allows the representation

$$Ax=Bx+Cx+Dx\,, \tag{6.7}$$

where B is a linear operator, C is a homogeneous operator of degree $s>1$ and D are terms of higher order than s. It is assumed that the operator B has a simple characteristic value $\lambda_0>0$ to which there corresponds a characteristic vector h_0: $Bh_0=\lambda_0 h_0$; every element $x\in E$ allows the unique representation of the form

$$x=Px+\xi(x)h_0\,,$$

where P is a linear operator of the projection onto the invariant subspace of the operator B, not containing h_0; $\xi(x)$ is a linear functional; the operator B commutes with P and

$$Bx=PBx+\lambda_0\xi(x)h_0 \qquad (x\in E)\,. \tag{6.8}$$

The operator C satisfies the conditions

$$C(tx)\equiv t^sCx\,,\ \|Cx_1-Cx_2\|\leqslant q_0r^{s-1}\|x_1-x_2\| \qquad (\|x_1\|,\ \|x_2\|\leqslant r)\,. \tag{6.9}$$

The operator D satisfies the conditions

$$\|Dx\|=o(\|x\|^s)\,,\quad \|Dx_1-Dx_2\|\leqslant q_1(r)\|x_1-x_2\|$$
$$(\|x_1\|,\ \|x_2\|\leqslant r,\ q_1(r)=o(r^{s-1}))\,. \tag{6.10}$$

The operator $B-\lambda_0 I$ has a continuous inverse in the subspace E_0.
The number λ_0 is in this case a point of bifurcation for the operator A. If the additional condition

$$\xi(Ch_0)\neq 0 \tag{6.11}$$

is satisfied, then a number of simple and important statements hold. We note that the inequality (6.11) describes the general case.

Theorem (6.12). *Suppose condition* (6.11) *is satisfied. Then numbers* $r_0>0$ *and* $\delta_0>0$ *can be found such that*
(1) *Equation* (6.3) *does not have non-zero solutions in the sphere* $\|x\|\leqslant r_0$

for $\lambda=\lambda_0$.

(2) *If s is even, then equation* (6.3) *has a unique continuous non-zero solution* $x(\lambda)$ *depending on* λ *in the sphere* $\|x\|\leqslant r_0$ *for* $|\lambda-\lambda_0|<\delta_0$ *where*

$$\operatorname{sign}\xi[x(\lambda)]=\operatorname{sign}\xi\,(Ch_0)\cdot\operatorname{sign}(\lambda-\lambda_0)\,, \tag{6.12}$$

$$\|x(\lambda)-\xi[x(\lambda)]h_0\|\leqslant M|\xi[x(\lambda)]|^s \tag{6.13}$$

and

$$M_1(\lambda)|\lambda-\lambda_0|^{1/(s-1)}\leqslant\|x(\lambda)\|\leqslant M_2(\lambda)|\lambda-\lambda_0|^{1/(s-1)}\,, \tag{6.14}$$

where

$$\lim_{\lambda\to\lambda_0}M_1(\lambda)=\lim_{\lambda\to\lambda_0}M_2(\lambda)=|\xi(Ch_0)|^{1/(s-1)}\,. \tag{6.15}$$

(3) *If s is odd and* $\xi(Ch_0)>0$ *then equation* (6.3) *does not have non-zero solutions in the sphere* $\|x\|\leqslant r_0$ *for* $\lambda\in(\lambda_0-\delta_0,\lambda_0)$ *and has exactly two continuous non-zero solutions* $x_1(\lambda)$ *and* $x_2(\lambda)$ *depending on* λ *for* $\lambda\in(\lambda_0,\lambda_0+\delta_0)$. *These solutions satisfy conditions* (6.13), (6.14) *and*

$$\operatorname{sign}\xi[x_1(\lambda)]>0,\quad\operatorname{sign}\xi[x_2(\lambda)]<0\,. \tag{6.16}$$

(4) *If s is odd and* $\xi(Ch_0)<0$ *then equation* (6.3) *does not have non-zero solutions in the sphere* $\|x\|\leqslant r_0$ *for* $\lambda\in(\lambda_0,\lambda_0+\delta_0)$ *and has exactly two continuous non-zero solutions* $x_1(\lambda)$ *and* $x_2(\lambda)$ *depending on* λ *for* $\lambda\in(\lambda_0-\delta_0,\lambda_0)$. *These solutions satisfy conditions* (6.13), (6.14) *and* (6.16).

The essential part of this proof of the theorem is completed with the establishment of the uniqueness of a non-zero solution, satisfying one of the conditions (6.16). It turns out that equation (6.3) can be replaced by an equivalent equation with an operator A_1 which is u_0-concave provided $\xi(Ch_0)<0$ and u_0-convex if $\xi(Ch_0)>0$ on some special image of the constructed cone. After this replacement uniqueness follows from general theorems.

Chapter 7. Applications

The general theory is applied to the investigation of those problems which can be reduced to equations with operators leaving some cone invariant. In this chapter the simplest examples of applications to several problems are given.

7.1. *Existence of positive solutions for integral equations*

We consider a linear integral operator

$$Ax(t)=\int_{\Omega} k(t, s)x(s)ds \tag{7.1}$$

with a non-negative kernel $k(t, s)$. The operator A leaves the cone K of non-negative functions invariant in that functional space in which it acts. Concerning all the linear integral operators A being considered in this section, it is assumed that they are u_0-positive and, moreover, that the inequalities

$$\alpha u_0(t)\leqslant Ax(t)\leqslant \beta u_0(t) \qquad (t\in\Omega) \tag{7.2}$$

are satisfied where α and β are positive for every non-negative function $x(t)$ which is not identically equal to zero, and $u_0(t)$ is some non-negative function. In order for condition (7.2) to be satisfied, it suffices that the kernel $k(t, s)$ satisfy the inequalities

$$u_0(t)\varphi(s)\leqslant k(t, s)\leqslant u_0(t)\psi(s) \qquad (t, s\in\Omega)\,, \tag{7.3}$$

where $\varphi(s)$ and $\psi(s)$ are non-negative, vanish only on sets of zero measure and have the properties that

$$\left|\int_{\Omega} x(s)\varphi(s)ds\right|<\infty\,, \quad \left|\int_{\Omega} x(s)\psi(s)ds\right|<\infty \qquad (x(t)\in E)\,.$$

The inequalities (7.3) can be replaced by a less restrictive but less easily verified condition (Lemma 7.1): there corresponds to every set $\Omega_1\subset\Omega$ of non-zero measure a number $\eta=\eta(\Omega_1)>0$ such that

$$\eta\int_{\Omega} k(t, s)ds\leqslant\int_{\Omega_1} k(t, s)ds \qquad (t\in\Omega)\,. \tag{7.4}$$

If the linear integral operator (7.1) is completely continuous in a functional space E and satisfies conditions (7.2), then there is a unique normalized non-negative characteristic function for it; to this characteristic function there corresponds a simple characteristic value which is strictly greater than the absolute value of all the remaining characteristic values. Other theorems of chapter 2 are valid for the operator (7.1).

Various questions about non-negative solutions of the non-linear integral equation

$$x(t)=\int_{\Omega} k[t, s, x(s)]ds \tag{7.5}$$

are studied.

It is assumed that the conditions are satisfied for which the integral operator

$$Ax(t)=\int_{\Omega} k[t, s, x(s)]\,ds \tag{7.6}$$

is completely continuous in some functional space E. For the case when E is the space C of continuous functions, complete continuity follows, for example, from the continuity of $k(t, s, u)$ with respect to the collection of variables. If the inequality

$$|k(t, s, u)| \leqslant k(t, s)(a+b|u|^{\alpha_0})$$

is satisfied where $\alpha_0 \geqslant 0$, $\int_{\Omega}\int_{\Omega}|k(t, s)|^{\beta_0}\,dt\,ds<\infty$ and $\alpha_0 \leqslant \beta_0-1$, then the operator (7.6) acts and is completely continuous in every space L_p where

$$p>1\,,\quad \frac{\alpha_0\beta_0}{\beta_0-1} \leqslant p \leqslant \beta_0\,.$$

Under natural assumptions the operator A is differentiable according to Fréchet (in the usual sense or with respect to a cone); its derivative $A'(\theta)$ is defined by the equality

$$A'(\theta)x(t)=\int_{\Omega} P(t, s)x(s)\,ds\,, \tag{7.7}$$

where

$$P(t, s)=k_u'(t, s, 0)\,.$$

If

$$\lim_{u\to+\infty}\frac{1}{u}k(t, s, u)=Q(t, s)\,,$$

then under further natural assumptions the operator (7.6) is strongly asymptotically linear with respect to the cone of non-negative functions where

$$A'(\infty)x(t)=\int_{\Omega} Q(t, s)x(s)\,ds\,. \tag{7.8}$$

The operator (7.6) leaves the cone of non-negative functions invariant if

$$k(t, s, u)\geqslant 0 \qquad (t, s\in\Omega,\ u\geqslant 0)\,. \tag{7.9}$$

If the further condition

$$\sup_{t\in\Omega} k(t, s, u)\leqslant \varrho \inf_{t\in\Omega} k(t, s, u) \qquad (s\in\Omega,\ u\geqslant 0) \tag{7.10}$$

is satisfied, then the operator (7.6) leaves a narrower cone $\tilde{K}$ of functions satisfying the inequality

$$\sup_{t\in\Omega} x(t) \leqslant \varrho \inf_{t\in\Omega} x(t) \tag{7.11}$$

invariant.

Various conditions for the existence of non-negative solutions for the equation (7.5) follow from the theorems of chapter 4.

If condition (7.9) is satisfied, then it suffices (Theorem 7.4) for the existence of at least one more non-negative solution for equation (7.5) that the kernel of the linear operator (7.8) does not have characteristic values which are greater than or equal to 1.

If $k(t, s, 0)\equiv 0$, then equation (7.5) has the trivial zero solution. In this connection, if the linear operators (7.7) and (7.8) exist and are u_0-positive (for example their kernels satisfy conditions (7.3) or (7.4)), then it suffices (Theorem 7.5) for the existence of a second non-negative solution that only the inequalities

$$\lambda_0 < 1 < \lambda_\infty ,$$

or inequalities

$$\lambda_\infty < 1 < \lambda_0 ,$$

are satisfied where λ_0 and λ_∞ denote the greatest characteristic values of the linear integral operators (7.7) and (7.8), respectively.

We consider a more particular family of equations

$$x(t) = \int_\Omega k(t, s) f[s, x(s)]\, ds \tag{7.12}$$

with a continuous positive kernel $k(t, s)$. If

$$f(s, 0)\equiv 0 , \quad f(s, u)\geqslant 0 , \quad f(s, u)\geqslant au^{1+\varepsilon_0} - b \qquad (u\geqslant 0) ,$$

where $a, b, \varepsilon_0 > 0$, then it suffices (Theorem 7.6) for the existence of a positive solution for equation (7.12) that the spectrum of the kernel $k(t, s) f_u'(s, 0)$ lie in a circle of radius $\varrho_0 < 1$.

The uniqueness of a non-negative solution for the equation (7.5) or equation (7.12) which is not identically equal to zero follows directly from the theorems about concave operators provided $k(t, s, u)$ or $f(t, u)$ (in the case of equation (7.5) or equation (7.12), respectively) is concave with respect to the variable u.

We can succeed in proving the uniqueness of a non-zero non-negative solution for equations with convex non-linearities only under rigorous restrictions (Theorem 7.8).

It follows from the theorems of chapter 5 about operators with monotone

minorants that the existence of a continuum of non-negative characteristic functions for the integral operator (7.6) is guaranteed by the inequality

$$k(t, s, u) \geqslant P(t, s)u \qquad (t,\ s \in \Omega,\ u \geqslant 0),$$

where the kernel $P(t, s)$ satisfies, for example, condition (7.3) or condition (7.4).

We do not linger over other applications of the general theory to non-linear integral equations.

7.2. *First boundary value problem for elliptic equations of second order with non-linearities*

Let Ω be a bounded open domain of N-dimensional space. The boundary Γ of the domain Ω is assumed to be sufficiently smooth. The differential expression

$$Lu(t) = -\sum_{i,j=1}^{N} a_{ij}(t)\frac{\partial^2 u}{\partial t_i \partial t_j} + \sum_{i=1}^{N} a_i(t)\frac{\partial u}{\partial t_i} + a(t)u \tag{7.13}$$

is considered where $t = \{t_1, \ldots, t_N\} \in \bar{\Omega}$, $\bar{\Omega} = \Omega + \Gamma$; it is assumed that

$$\sum_{i,j=1}^{N} a_{ij}\xi_i\xi_j \geqslant \gamma \sum_{i=1}^{N} \xi_i^2 \qquad (\gamma > 0)$$

for arbitrary $\xi_1, \ldots, \xi_N$ and the function $a(t)$ is non-negative. The coefficients of the differential expression (7.13) are assumed to be smooth.

The Green function of the differential operator L is non-negative for zero boundary conditions and satisfies condition (7.4). This allows the application of general theorems from chapters 4–6 to the investigation of the first boundary value problem for equations with the operator (7.13).

Theorem (7.10). *The boundary value problem*

$$\lambda Lu(t) = u(t), \quad u(t)|_{t \in \Gamma} = 0 \tag{7.14}$$

has a unique normalized non-negative characteristic function to which there corresponds a positive characteristic value λ_0. The characteristic value λ_0 is simple. If other characteristic values exist then their absolute magnitudes are strictly less than λ_0.

We now consider the non-linear boundary value problem

$$\lambda Lu(t) = f\left(t, u, \frac{\partial u}{\partial t_1}, \ldots, \frac{\partial u}{\partial t_N}\right), \qquad u(t)|_{t \in \Gamma} = 0. \tag{7.15}$$

We shall assume that the function

$$f(t, u, v) \equiv f(t_1, \ldots, t_N, u, v_1, \ldots, v_N)$$

is continuous with respect to the collection of variables in the domain $\tilde{\Omega}(t \in \bar{\Omega}, u \geqslant 0, -\infty < v_1, \ldots, v_N < \infty)$ and assumes non-negative values in this domain.

It follows from the theorems about operators with monotone minorants that it suffices for the existence of a continuum of non-negative solutions for the boundary problem (7.15) corresponding to some values of the parameter λ that the condition

$$f(t, u, v) \geqslant \alpha u \qquad (\{t, u, v\} \in \Omega\,,\ \alpha > 0) \tag{7.16}$$

be satisfied.

We turn to the problem

$$Lu(t) = f\left(t, u, \frac{\partial u}{\partial t_1}, \ldots, \frac{\partial u}{\partial t_N}\right), \qquad u(t)|_{t \in \Gamma} = 0\,. \tag{7.17}$$

If

$$f(t, 0, 0, \ldots, 0) \equiv 0 \qquad (t \in \Omega)\,,$$

then the problem (7.17) has the trivial zero solution. It follows from the theorems about operators compressing the cone that it suffices (Theorem 7.12) for the existence of a non-trivial non-negative solution that the following conditions be satisfied:

(a) The function $f(t, u, v)$ satisfies the inequality

$$0 \leqslant f(t, u, v) \leqslant a + bu \qquad (a, b \geqslant 0) \tag{7.18}$$

in the domain $\tilde{\Omega}$ where $b\lambda_0 < 1$, and λ_0 is the greatest characteristic value of the boundary value problem (7.14); b) the greatest characteristic value of the linear boundary value problem

$$\lambda Lu = f_u'(t, 0, 0)u\,, \quad u(t)|_{t \in \Gamma} = 0$$

satisfies the condition $\Lambda(\theta) > 1$.

We can obtain a closely related theorem about the conditions for the existence of a non-trivial non-negative solution as a corollary to the theorems about operators expanding the cone (see Theorem 7.13).

Finally, we consider a simpler boundary value problem

$$Lu = f(t, u)\,, \quad u(t)|_{t \in \Gamma} = 0\,. \tag{7.19}$$

If the function $f(t, u)$ does not decrease with respect to the variable u and is concave, then this problem reduces to an equation with a u_0-

concave operator. From here follow various statements about the uniqueness of a non-trivial non-negative solution. For the convergence to it of the successive approximations, see Theorems 7.14 and 7.15.

7.3. *Existence of positive periodic solutions for a system of ordinary differential equations of first order*

A system of ordinary differential equations

$$\begin{aligned} \frac{dx_1}{dt} &= f_1(t, x_1, \ldots, x_n), \\ &\cdots\cdots\cdots\cdots \\ \frac{dx_n}{dt} &= f_n(t, x_1, \ldots, x_n) \end{aligned} \tag{7.20}$$

is studied; it is assumed that the first members are periodic with period ω with respect to t and are sufficiently smooth.

We assume that every function $f_i(t, x_1, \ldots, x_n)$ satisfies the condition

$$f_i(t, x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) \geqslant 0 \qquad (x_j > 0 \text{ for } j \neq i). \tag{7.21}$$

Then every point of the cone K, composed of vectors with non-negative components, under movement with respect to the trajectories of system (7.20) does not fall outside of the cone K. We assume that every trajectory is extendible to the interval of time which is equal to ω (this essential restriction is not involved in definite formulations), and we define the transformation A of the cone K into itself by the following rule: Ax is the point into which the point x goes under motion with respect to the trajectory of the system in the time ω. Every fixed point of the transformation A of the cone K into itself defines, obviously, a periodic solution of system (7.20).

The described plan allows the obtaining of criteria from the general theorems of chapters 4–6 for the existence of positive periodic solutions. We give one example.

We assume that the system (7.20) has the trivial zero solution. We introduce the notation

$$b_{ij}(t) = \frac{\partial}{\partial x_j} f_i(t, 0, \ldots, 0) \qquad (i, j = 1, \ldots, n)$$

and consider the linear system

$$\frac{dx_1}{dt} = b_{11}x_1 + \ldots + b_{1n}x_n,$$
$$\ldots\ldots\ldots\ldots\ldots \tag{7.22}$$
$$\frac{dx_n}{dt} = b_{n1}x_1 + \ldots + b_{nn}x_n.$$

We shall assume that the functions $b_{ij}(t)$ are postitive for $i \neq j$. The coefficients of system (7.22) are periodic with period ω with respect to t. We denote by λ_0 the greatest characteristic value of the matrix of monodromy of the system (7.22).

It turns out that it suffices for the existence of a non-trivial periodic solution for system (7.20) that either the condition

$$\lambda_0 > 1, \quad \sum_{i=1}^{n} x_i f_i(x_1, \ldots, x_n) \leqslant 0 \text{ for } x_j \geqslant 0 \text{ and large } \sum_{i=1}^{n} x_i$$

or the condition

$$\lambda_0 < 1, \quad \sum_{i=1}^{n} x_i f_i(x_1, \ldots, x_n) \geqslant 0 \text{ for } x_j \geqslant 0 \text{ and large } \sum_{i=1}^{n} x_i$$

be satisfied. The first of these conditions corresponds to the operators compressing the cone, and the second corresponds to those expanding the cone.

If the system (7.20) behaves as a linear system for large values of the variables and if the greatest characteristic value of the matrix of monodromy of this linear system is denoted by λ_∞ then the conditions can be formulated for the existence of a non-trivial non-negative periodic solution in a simple manner. It suffices (Theorem 7.20) that either the inequalities

$$\lambda_0 < 1 < \lambda_\infty$$

or the inequalities

$$\lambda_\infty < 1 < \lambda_0$$

be satisfied.

7.4. *Two point boundary value problem*

Various questions which are connected with non-negative solutions of the problem

$$\frac{d^2x}{dt^2} + f\left(t, x, \frac{dx}{dt}\right) = 0, \quad x(0) = x(1) = 0 \tag{7.23}$$

are studied where the function $f(t, x, y)$ is continuous with respect to the

collection of variables. After the substitution

$$-\frac{d^2x}{dt^2} = v(t), \quad x(0)=x(1)=0,$$

the problem (7.23) reduces to the equation

$$v(t)=f\left[t, \int_0^1 G(t,s)v(s)\,ds, \quad \int_0^1 G_t'(t,s)v(s)\,ds\right], \tag{7.24}$$

where

$$G(t,s) = \begin{cases} t(1-s), & \text{if} \quad t\leqslant s, \\ s(1-t), & \text{if} \quad s\leqslant t. \end{cases} \tag{7.25}$$

The operator F, defined by the first member of the equation (7.24), is completely continuous in the space C of functions which are continuous on [0, 1].

If the condition

$$f(t,x,y)\geqslant 0 \qquad (t\in[0,1],\ x\geqslant 0,\ -\infty<y<\infty)$$

is satisfied, then the operator F leaves the cone K of non-negative functions invariant. This allows the application of the results of chapters 4–6 to the investigation of problem (7.23).

We formulate here one result (which, however, can be obtained without conversion to integral equations).

We c onsider the case when the problem (7.23) has the trivial zero solution. Let

$$a(t)=f_x'(t,0,0), \quad \lim_{x\to+\infty} \sup_{t,\,y} \left|\frac{1}{x}f(t,x,y)-b(t)\right|=0.$$

It follows from the theorems about operators which expand or compress a cone that the problem (7.23) has at least one non-trivial non-zero solution if every non-trivial solution of one of the linear equations

$$\frac{d^2x}{dt^2}+a(t)x=0, \quad \frac{d^2x}{dt^2}+b(t)x=0$$

does not vanish twice on (0, 1) and the other of the indicated equations has at least one solution which vanishes twice on (0, 1).

The two point problem for the system

$$\begin{aligned} &\frac{d^2x_1}{dt^2}+f_1\left(t, x_1, \ldots, x_n, \frac{dx_1}{dt}, \ldots, \frac{dx_n}{dt}\right)=0, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\frac{d^2x_n}{dt^2}+f_n\left(t, x_1, \ldots, x_n, \frac{dx_1}{dt}, \ldots, \frac{dx_n}{dt}\right)=0 \end{aligned} \tag{7.26}$$

is contained in the determination of the solutions which satisfy the boundary conditions

$$x_1(0)=x_2(0)=\ \dots\ =x_n(0)=x_1(1)=x_2(1)=\ \dots\ =x_n(1)=0\,. \tag{7.27}$$

Methods developed in chapters 4–6 can also be applied for the investigation of this problem if

$$f_i(t, x_1, \dots, x_n, y_1, \dots, y_n)\geqslant 0 \tag{7.28}$$
$$(t\in[0, 1],\ x_1, \dots, x_n\geqslant 0,\ -\infty<y_1, \dots, y_n<\infty,\ i=1, \dots, n)\,.$$

7.5. *Periodic solutions of systems of second order*

If the function $f(t, x, y)$ is periodic with period 2 with respect to t and odd with respect to t and x simultaneously:

$$f(t+\omega, x, y)\equiv f(t, x, y),\ f(-t, -x, y)\equiv -f(t, x, y)\,, \tag{7.29}$$

then every solution $x(t)$ of the two point problem (7.23) can be extended to a periodic solution of the equation

$$\frac{d^2x}{dt^2}+f\left(t, x, \frac{dx}{dt}\right)=0\,. \tag{7.30}$$

To this end it suffices to extend the solution from the interval [0, 1] to the interval $[-1, 1]$ as an odd function, and thus to the remaining values t as a periodic function. Thus, every theorem on the existence of a solution for a two point problem (7.23) or (7.26)–(7.27) can be treated as a theorem on the existence of a periodic solution for a corresponding equation or system of equations.

Specific difficulties arise in the study of self-regulating systems. These difficulties are caused by the fact that the period is unknown a priori.

We first consider a frictionless system

$$\begin{aligned}&\frac{d^2x_1}{dt^2}+g_1(x_1, \dots, x_n)=0\,,\\ &\dots\dots\dots\dots\dots\dots\\ &\frac{d^2x_n}{dt^2}+g_n(x_1, \dots, x_n)=0\,.\end{aligned} \tag{7.31}$$

We make the substitution

$$t=\omega\tau\,. \tag{7.32}$$

Then the system (7.31) is written in the form

$$\frac{d^2x_1}{d\tau^2} + \omega^2 g_1(x_1, \ldots, x_n) = 0 ,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots \tag{7.33}$$

$$\frac{d^2x_n}{d\tau^2} + \omega^2 g_n(x_1, \ldots, x_n) = 0 .$$

It is obvious that the substitution (7.32) establishes a one-to-one correspondence between the periodic solutions with the period 2ω of the system (7.31) and the periodic solutions with the period 2 of the system (7.33). If the functions $g_i(x_1, \ldots. x_n)$ are odd with respect to the collection of variables:

$$g_i(-x_1, \ldots, -x_n) \equiv -g_i(x_1, \ldots, x_n) \qquad (x_j \geqslant 0,\ i, j = 1, \ldots, n) , \tag{7.34}$$

then we can prove the existence of solutions for the two point problem for the proof of the existence of periodic (with period 2) solutions for the system (7.33). The question about the solutions of a two point problem reduces to an investigation of a system of integral equations

$$\int_0^1 G(t, s) g_1[x_1(s), \ldots, x_n(s)]\, ds = \frac{1}{\omega^2} x_1(s) ,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots \tag{7.35}$$

$$\int_0^1 G(t\ s) g_n[x_1(s)\ \ldots, x_n(s)]\, ds = \frac{1}{\omega^2} x_n(s) ,$$

where $G(t, s)$ is the function (7.25). The system (7.35) can be considered as a problem about characteristic functions of some integral operator in the space of continuous vector-functions. This allows the application, for example, of the theorem about operators with monotone minorants. The arguments given reduce, in particular, to the following statement:

Theorem (7.24). *Let* $g_i(x_1, \ldots, x_n) \geqslant 0$ *for* $x_j \geqslant 0$ $(i, j = 1, \ldots, n)$ *and suppose the functions* $g_i(x_1, \ldots, x_n)$ *satisfy the condition* (7.34) *and let at least one of the functions* $g_i(x_1, \ldots, x_n)$ *satisfy the inequality*

$$g_{i_0}(x_1, \ldots, x_n) \geqslant k x_{i_0} \qquad (x_j \geqslant 0,\ j = 1, \ldots, n,\ k > 0) .$$

Then the system (7.31) *has a continuum of distinct periodic solutions.*

We now consider a system of general form:

$$
\begin{aligned}
-\frac{d^2x_1}{dt^2} &= f_1\left(x_1, \ldots, x_n, \frac{dx_1}{dt}, \ldots, \frac{dx_n}{dt}\right), \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
-\frac{d^2x_n}{dt^2} &= f_n\left(x_1, \ldots, x_n, \frac{dx_1}{dt}, \ldots, \frac{dx_n}{dt}\right).
\end{aligned} \tag{7.36}
$$

The substitution (7.32) reduces to a more complicated system than (7.35):

$$
\begin{aligned}
-\frac{d^2x_1}{d\tau^2} &= \omega^2 f_1\left(x_1, \ldots, x_n, \frac{1}{\omega}\frac{dx_1}{d\tau}, \ldots, \frac{1}{\omega}\frac{dx_n}{d\tau}\right), \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
-\frac{d^2x_n}{d\tau^2} &= \omega^2 f_n\left(x_1, \ldots, x_n, \frac{1}{\omega}\frac{dx_1}{d\tau}, \ldots, \frac{1}{\omega}\frac{dx_n}{d\tau}\right).
\end{aligned}
$$

However, the general results of chapters 4–6 are also applicable here. For example, if the right members of the equations (7.36) are odd with respect to the collection of variables $x_1, \ldots, x_n$, then it suffices (Theorem 7.25) for the existence for the system (7.36) of a continuum of distinct periodic solutions that the inequalities

$$
\begin{aligned}
&0 \leqslant f_i(x_1, \ldots, x_n, y_1, \ldots, y_n) \leqslant k_1(x_1 + \ldots + x_n) + k_2 \\
&(x_j \geqslant 0, \quad -\infty < y_k < \infty, \quad i, j, k = 1, \ldots, n)
\end{aligned}
$$

be satisfied and that a number $\delta > 0$ be found such that

$$
\begin{aligned}
&f_i(x_1, \ldots, x_n, y_1, \ldots, y_n) \geqslant k_0(x_1 + \ldots + x_n) \\
&(0 \leqslant x_j \leqslant \delta, \quad -\infty < y_k < \infty, \quad i, j, k = 1, \ldots, n),
\end{aligned}
$$

where $k_1 < k_0$. Under the conditions of this theorem the system (7.36) has periodic solutions of all periods 2ω, where ω satisfies the inequalities

$$
\frac{\pi}{\sqrt{nk_0}} < \omega < \frac{\pi}{\sqrt{nk_1}}.
$$

7.6. *Dirichlet problem for the Monge-Ampère equation*

We denote by Ω a bounded convex domain in the $\{x, y\}$-plane; Γ is the boundary of the domain Ω. We assume that the specific curvature of the curve Γ is bounded below.

The question about the existence of solutions of the equation

$$
\frac{rt - s^2}{(1 + p^2 + q^2)^\alpha} = f(x, y, z, p, q) \qquad (0 \leqslant \alpha \leqslant 1) \tag{7.37}
$$

satisfying the zero boundary condition

$$z(x, y)|_\Gamma = 0 \tag{7.38}$$

is considered.

It is assumed that the function $f(x, y, z, p, q)(\{x, y\} \in \Omega + \Gamma, z \geqslant 0, -\infty < p, q < \infty)$ is continuous with respect to the collection of variables and is non-negative and uniformly bounded provided the values z belong to a finite interval.

It is clarified that the boundary value problem (7.37)–(7.38) is equivalent to the operator equation

$$z(x, y) = \boldsymbol{B}z(x, y) = \boldsymbol{A}_\alpha f[x, y, z(x, y), z_x(x, y), z_y(x, y)], \tag{7.39}$$

where $\boldsymbol{A}_2$ is the inversion of the Monge-Ampère operator $(rt - s^2) \times (1 + p^2 + q^2)^{-\alpha}$ under the boundary condition (7.38).

The operator $\boldsymbol{B}$ transforms the cone K of non-negative convex functions satisfying the condition (7.38) into itself. This allows application of general principles to the investigation of equation (7.39) (that is, to the investigation to the boundary value problem (7.37)–(7.38)).

It is clarified that *the satisfaction of the conditions*

$$0 \leqslant f(x, y, z, p, q) \leqslant \begin{cases} a_1(1 + z^2)^\gamma, & \text{if } 0 \leqslant \alpha < 1, \\ a_1 \ln^{1-\varepsilon}(2 + z) & \text{if } \alpha = 1 \end{cases} \tag{7.40}$$

suffice for the existence of at least one solution for the boundary value problem (7.37)–(7.38) (Theorem 7.26).

We further assume that $f(x, y, 0, 0, 0) \equiv 0$. Then the boundary value problem has the trivial zero solution. It suffices (Theorem 7.27) for the existence of still one more non-trivial solution that the inequality

$$f(x, y, z, p, q) \geqslant a_2 z^{2-\varepsilon} \qquad (0 \leqslant z \leqslant \delta_0) \tag{7.41}$$

be satisfied where a_2, ε and δ_0 are some positive numbers besides condition (7.40).

Theorem (7.28). *If* $f(x, y, z, p, q) \equiv f(x, y, z)$ *and* $0 \leqslant \alpha < 1$ *where* $f(x, y, z)$ *does not decrease with respect to* z *and is homogeneous with respect to* z *of degree* $\gamma < 2(1 - \alpha)$, *then the boundary value problem* (7.37)–(7.38) *has not more than one non-trivial solution.*

It is of interest to note that this theorem on the uniqueness of a non-trivial solution is obtained under conditions when there is no maximum principle.

In the above formulated statements it was assumed that $f(x, y, z, p, q)$ increases essentially slower than z^2 with respect to the variable z. We

give one criterion for the existence of a non-zero solution comprising the case of "strong" non-linearities with respect to z.

Theorem (7.29). *Let numbers* $\delta_0>0$ *and* $M_0>0$ *be found such that*

$$f(x, y, z, p, q)\leqslant a_3 z^{\gamma_1} \qquad (0\leqslant z\leqslant \delta_0)$$

and

$$f(x, y, z, p, q)\geqslant a_4 z^{\gamma_2} \qquad (z\geqslant M_0)\,,$$

where $\gamma_1, \gamma_2>2$, $a_3, a_4>0$. *Then the boundary value problem* (7.37)–(7.38) *has at least one more solution besides the trivial zero solution.*

We deduce that conditions for the existence for the boundary problem being considered of many (and even an infinite number) of solutions are found without difficulty. To this end it suffices that the function $f(x, y, z, p, q)$ have alternating parts of fast and slow growth with respect to the variable z.

BIBLIOGRAPHICAL NOTES

A small number of papers are well known in which problems specifically connected with positive solutions of distinct non-linear equations are studied by classical methods. We note here the article of P. S. Uryson [1] which is concerned with the consideration of a class of non-linear integral equations. This article of P. S. Uryson was little known for several years but after its republication in 1949 (P. S. Uryson [2]) it attracted attention and served as a starting point for many investigations.

The methods of functional analysis which are based on the use of partially ordered spaces whose theory is connected with the names of F. Riesz, M. G. Kreĭn, L. V. Kantorovič, G. Freidental, G. Birkhoff and others are a natural tool for the investigation of positive solutions. The methods of partially ordered spaces were applied by many authors (the list of papers cited at the end of this book does not claim completeness) to problems about positive solutions in different aspects.

In the present book the theory of partially ordered spaces is used in that geometrical treatment of spaces with cones which was developed by M. K. Kreĭn and his students.

We note here that the first results involving completely continuous operators leaving the cone invariant were obtained by M. A. Rutman.

The methods discussed in this book for the investigation of positive solutions of non-linear operator equations were developed basically by the author and his students L. A. Ladyženskii, I. A. Bakhtin and V. Ya. Stecenko; isolated results are published for the first time (see further references to corresponding papers further on).

Chapter 1. Spaces with a cone

1.1. In the study of fundamental concepts involving spaces with a cone we follow M. G. Kreĭn (see M. G. Kreĭn and M. A. Rutman [1]). Before the war a manuscript was prepared by M. G. Kreĭn on the theory of

cones; unfortunately, this manuscript was not published. In particular, statements were contained in the manuscript of M. G. Kreĭn which are closely related to many facts which are studied in the first chapter.

1.2. The concept of a normal cone was introduced by M. G. Kreĭn. The criterion for normality of a solid cone was found by D. P. Milman (see M. G. Kreĭn and M. A. Rutman [1]). M. G. Kreĭn found the criterion for the normality of a cone in the properties of a conjugate semigroup. I. A. Bakhtin [2, 4] found the criterion for the normality of a cone contained in Theorem 1.2. An approximate criterion for normality was established earlier by Ya. I. Grossberg and M. G. Kreĭn [1]. Theorem 1.1 in the form in which it was cited is found in the article of M. A. Krasnosel'skii [8].

1.3. Apparently, for the first time the u_0-norm was introduced and studied by M. G. Kreĭn (see N. I. Akhiezer and M. G. Kreĭn [1]). The u_0-norm plays an essential role in several papers of Leningrad mathematicians (see L. V. Kantorovič, B. Z. Vulikh, A. G. Pinsker [1]). The u_0-norm was applied and studied by other authors (see, for example, M. G. Kreĭn and S. G. Kreĭn [1], I. A. Bakhtin and M. A. Krasnosel'skii [2]).
We note that M. G. Kreĭn called the cone *acute* provided the norm is monotone on it.

1.4. The general theory of linear positive functionals in Banach spaces with a cone is developed by M. G. Kreĭn; in the book only those facts of the theory were cited which are used in the subsequent constructions.
The concepts of a uniformly positive functional and cone which allows plastering were introduced by M. A. Krasnosels'kii [8]. However, the property of the cone to allow plastering (without isolating this property as a special concept) was used earlier.

1.5. Regular and completely regular cones are singled out in an article of M. A. Krasnosel'skii [8]; in this article Theorems 1.6–1.8 are cited. Theorem 1.9 is proved by V. Ya. Stecenko.

1.6. Theorems 1.10–1.13 were cited earlier in the article of M. A. Krasnosel'skii [8]. The constructions of subsection 3 use special functional spaces isolated by W. Orlicz [1, 2]. The properties used in these constructions of the norms of elements of the spaces were established earlier simultaneously by M. A. Krasnosel'skii and Ya. B. Rutickii [1].

1.7. The concept of a minihedral cone is due to M. G. Kreĭn. The

spaces with minihedral and strongly minihedral cones are different classes of KB-spaces of L. V. Kantorovič; the theory of such spaces is studied in detail in the books of L. V. Kantorovič, B. Z. Vulikh and A. G. Pinsker [1] and B. Z. Vulikh [1]. Examples at the end of this section were suggested by V. Ya. Stecenko; to him belongs the presentation of the proof of Theorem 1.14 (an analogous proof is presented in the book of L. V. Kantorovič, B. Z. Vulikh and A. G. Pinsker).

1.8. The basic results of this section belong to V. Ya. Stecenko. Theorem 1.6 is due to I. A. Bakhtin [4]. The concepts of K-regular and fully K-regular cones were suggested by M. A. Krasnosel'skii.

Chapter 2. Linear positive operators

2.1. The class of u_0-positive operators was isolated and studied by M. A. Krasnosel'skii and L. A. Ladyženskii (see M. A. Krasnosel'skii and L. A. Ladyženskii [2], M. A. Krasnosel'skii [4, 5]). Theorem 2.2 was formulated as a hypothesis by M. A. Krasnosel'skii and proved by V. Ya. Stecenko; in this book a new proof is given. The remaining results were not indicated earlier. We note that Theorem 2.3 can be considered as a generalization of the theorem of M. G. Kreĭn for the continuity of a positive functional on a solid cone (the subsequent generalization is given in the article of I. A. Bakhtin, M. A. Krasnosel'skii and V. Ya. Stecenko [1]).

2.2. In a n-dimensional real space with a cone K of vectors with non-negative coordinates the positive linear operator is a matrix with non-negative elements. The existence for such a matrix of characteristic vectors with non-negative coordinates was established by Perron. P. S. Aleksandrov and G. Hopf have shown that the Perron theorem can be proved by means of Brouwer's fixed point topological theorem.
Some statements about matrices with non-negative elements which supplement the Perron theorem are cited in the book of F. P. Gantmakher [1].
An analogy of the Perrson theorem for the case of linear integral operators

$$Ax(t)=\int_{\Omega} k(t, s)\,x(s)\,ds$$

is the Jentzsch theorem [1] in which it is stated that a linear integral operater A has a positive characteristic function provided the kernel $k(t, s)$

is continuous and positive. In the development of the concept of P. S. Aleksandrov and G. Hopf the proof of the Jentzsch theorem was obtained by M. A. Rutman [1, 2]. This result allowed transition to a more detailed analysis of linear positive operators. Here, the basic results were obtained by M. G. Kreĭn.

The existence of characteristic vectors in the cone K for operators leaving this cone invariant is proved under various assumptions in the article of M. G. Kreĭn and M. A. Rutman [1]. In particular, the theorem proved in this article for the existence of a characteristic positive vector for a completely continuous operator is closely related to Theorem 2.5 (in this book another proof is given; compare M. A. Krasnosel'skii [5]). The proof of Theorem 2.9 is given in the article of M. G. Kreĭn and M. A. Rutman. Theorem 2.6 is closely related to a theorem of E. Rothe.

In subsection 2.2.6 operators are considered in weakly complete spaces. Theorems 2.7 and 2.8 allow generalization. A weak topology can be considered in the space E generated not by all the linear functionals but by some linear set E_0^* of functionals. It is then necessary to require of the cone K that on it there can be defined a uniformly positive function $f(x) \in E_0^*$. The operator A must be weakly continuous in that weak topology which is defined by the set E_0^* of linear functionals.

Thus, for example, if E is the space of linear functionals on some separable Banach space E_1 then E_0^* can be considered as the set of those functionals $f(x)$ which are representable in the form

$$f(x) = x(\varphi_0),$$

where φ_0 is some fixed point of the space E_1. The space E_1 in this case is weakly complete and weakly compact. If K consists of functionals which are positive on a solid cone $K_1 \subset E_1$, then all the functionals $f(x) = x(\varphi_0)$ will be uniformly positive on the entire cone K where φ_0 is an interior element of the cone K_1. In the case being considered, A is continuous in the weak topology generated by the set E_0^* of linear functionals provided A is the linear operator conjugate to some operator acting in the space E_1. The last case which is described was closely studied by M. G. Kreĭn (see also M. G. Kreĭn and M. A. Rutman [1]); the arguments of M. G. Kreĭn are presented as a basis for the proof of Theorem 2.7.

Collections of positive operators, commuting one with the other in a conjugate space, were also studied by M. G. Kreĭn; for such operators there are, generally speaking, a common characteristic vector. This refined

fact contains a number of important statements of the type of the theorem of N. N. Bogolyubov and N. M. Krylov concerning the existence of invariant measures in compact dynamical systems.
The role of conjugate operators for non-linear operators with which the book is concerned is not large at the present time. Therefore, the results of M. G. Kreĭn though important in other problems are not studied here.
A number of propositions concerning characteristic vector operators which leave two cones K and $K_0(K_0 \subset K)$ invariant were established by I. A. Bakhtin [3, 4].

2.3,4 The investigation of the multiplicity of a positive characteristic value and the comparsion of this characteristic value with the remaining values for the case of a matrix with positive elements was carried out by Perron, and for integral operators with positive kernels, by Jentzsch. Analogous results are investigated in the article of M. G. Kreĭn and M. A. Rutman [1] for operators A which leave the solid cone K invariant and are strongly positive in the sense that $A^n x (x \in K, x \neq \theta)$ is an interior element of the cone K for some number n.
The corresponding theory for u_0-positive operators is investigated in the book of M. A. Krasnosel'skii [5]. Here essential supplements and simplifications are given. In this connection, in particular, some considerations of I. A. Bakhtin [4] are used.
Theorem 2.12 was indicated in the article of M. A. Krasnosel'skii [8].

2.5. Non-homogeneous linear equations with positive integral operators were subjected to special analysis apparently for the first time by P. S. Uryson [1] who established propositions which are the simplest analogy of Theorem 2.16. L. A. Ladyženskii [2] and I. A. Bakhtin [3, 4] considered non-homogeneous equations with abstract positive operators.
Theorem 2.16 was indicated by M. A. Krasnosel'skii. The theorems proved in subsection 2.5.4 are the development of lemmas concerning the estimate of the coefficient (M. A. Krasnosel'skii [4, 5]).
Subsection 2.5.2 was written in collaboration with A. Ya. Levin.

Chapter 3. Differentiability with respect to a cone

3.1. The basic facts of the differential calculus in Banach spaces are investigated in the courses of L. A. Lyusternik and V. I. Sobolev [1] and

L. V. Kantorovič and G. P. Akilov [1]. For a more detailed investigation see the book of Hille [1].

Concepts of derivatives with respect to a cone were proposed by M. A. Krasnosel'skii. The derivatives with respect to a cone of some integral operators are found in an article of M. A. Krasnosel'skii and L. A. Ladyženskii [3].

In connection with Theorems 3.1 and 3.2, the question concerning the coincidence of the class of reproducing and the class of non-oblate cones was raised in a seminar by M. A. Krasnosel'skii. The positive reply to this question was obtained by I. A. Bakhtin (its proof is reproduced in this book). As M. G. Kreĭn informed us, the non-oblateness of some classes of reproducing cones was established by him already in 1940, and then in general form by V. L. Smul'yan. The proof of V. L. Smul'yan consists of the establishment of the completeness of the space E with respect to the norm

$$\|x\|_1 = \inf_{u,v\in K, u-v=x} (\|u\| + \|v\|)$$

and in the subsequent use of Banach's theorem concerning the continuity of the inverse operator.

3.2. The operators differentiable at infinity were isolated as a special class apparently for the first time by M. A. Krasnosel'skii [2] (see also M. A. Krasnosel'skii [5]). The calculation of strong asymptotic operators is carried out in the article of M. A. Krasnosel'skii and L. A. Ladyženskii [3].

3.3,4. Part of the theorems of these sections was cited in an article of M. A. Krasnosel'skii [9]. Recently, Ya. V. Pokornii strengthened Theorems 3.7, 3.9 and 3.12 eliminating the assumption about complete continuity (or weak continuity) of the operators being studied.

Chapter 4. Existence of positive solutions

4.1. Theorem 4.1 under condition 4.1a was established by Birkhoff [1]. Convergence of successive approximations to the solution under conditions 4.1b and 4.1c was used by many authors and is a well-known fact. A more general statement than Theorem 4.1 and Theorem 4.2 is proved in an article of I. A. Bakhtin and M. A. Krasnosel'skii [3]; this more general statement contains Theorem 4.3.

The expression "lemma about two militiamen" comes from student folklore!

4.2–4. Part of the results was indicated in the article of M. A. Krasnosel'skii [9].

Chapter 5. Continuous branches of positive solutions

5.1. The concept of a continuous branch was introduced by M. A. Krasnosel'skii [4].
The simplest theorem on the existence of a characteristic vector for an abstract non-linear positive operator was proved by E. Rothe [1]. Also, some simple theorems on the existence of characteristic non-negative functions for non-linear operators were obtained by M. A. Rutman (see M. G. Kreĭn and M. A. Rutman [1], section 5.1.9).
The theorems investigated in this section were obtained in the main by M. A. Krasnosel'skii and published earlier.

5.2. Concerning basic topological concepts, see P. S. Aleksandrov [1] or L. S. Pontrjagin [1] or M. A. Krasnosel'skii [5].
The proof of Theorem 5.5 in the main repeats an appendix of M. A. Krasnosel'skii to an article of M. A. Krasnosel'skii and L. A. Ladyženskii [2].

5.3. In the main, theorems published earlier by the author are repeated here.

Chapter 6. Equations with concave operators

6.1. Basic concepts and facts of the theory of concave operators were introduced and established by M. A. Krasnosel'skii and L. A. Ladyženskii (see M. A. Krasnosel'skii and L. A. Ladyženskii [2], M. A. Krasnosel'skii [5]). The theory of concave operators obtained an essential supplementation and improvement in a dissertation of I. A. Bakhtin [4] (see also I. A. Bakhtin [1, 5]). In particular, a family of operators was isolated by him which are considered in Theorem 6.5; a family of operators concave on some cone in the sense of a partial ordering generated by a second cone was subjected by him to special investigation. In the present

book, the theory of concave operators obtained an additional development; the definitions are somewhat changed.
Theorems 6.6 and 6.7 were published earlier (I. A. Bakhtin and M. A. Krasnosel'skii [2, 3]).

6.2. The results of subsections 6.2.1, 6.2.3 and 6.2.4 are close to those published earlier (M. A. Krasnosel'skii and L. A. Ladyženskii [2], M. A. Krasnosel'skii [5]).
For the proof of the existence of characteristic vectors for concave completely continuous operators, I. A. Bakhtin proposed a special device. As usual, we first consider the operators $A_n x = Ax + (v/n)$ where v is a suitably chosen element. Since $A\theta \neq \theta$, then the existence of characteristic vectors is successfully proved by the method of successive approximations; moreover, the existence of a continuous branch of characteristic vectors for concave operators A_n is proved. Thus, the boundedness below is proved for the numbers λ_n in the equalities $A_n x_n = \lambda_n x_n (\|x_n\| = \text{const.})$, and the characteristic vector of the operator A is constructed as a limit of some subsequence x_{n_i}. We prefer to retain the proof based on topological concepts (in the case of a concave operator, it suffices to apply Schauder's principle).

6.3. The attempt to isolate families of convex operators, for which the uniqueness theorem is valid for a positive non-zero solution, was undertaken on my proposal by L. A. Ladyženskii and I. A. Bakhtin. They both independently obtained propositions close to Lemma 6.3.
L. A. Ladyženskii [2] indicated a number of examples of convex operators for which the uniqueness theorem is not true. The investigation of I. A. Bakhtin [4] of equations with convex operators contains a number of refined considerations; however, the results which were obtained by them were little effective and far from conclusive.
The uniqueness principle mentioned in the book for equations with convex operators is indicated by M. A. Krasnosel'skii in connection with the study of a special family of integral equations with convex non-linearities (see chapter 7, section 7.1).

6.4. The basic Theorem 6.12 is due to M. A. Krasnosel'skii.
Equation (6.70) is the equation of a branching appearing in the theory of Lyapunov-Schmidt for the branching of small solutions. We note that for some additional assumptions concerning the smoothness of the oper-

ator being considered Theorem 6.12 can be proved in an easier fashion. However, it appears to us that the proof given in this book is of interest in itself.

Lemmas 6.5–6.7 are cited and proved for one particular case in the articles of I. A. Bakhtin and M. A. Krasnosel'skii [2, 3].

Chapter 7. Applications

7.1. Concerning conditions for complete continuity of linear integral operators for various functional spaces, see general textbooks on functional analysis (S. Banach [1], L. A. Lyusternik and V. I. Sobolev [1], L. V. Kantorovič and G. P. Akilov [1], and others). Necessary and sufficient conditions for the complete continuity of a linear integral operator in the space C were found by Radon [1]. Conditions for the complete continuity of linear integral operators in Orlicz spaces are seen in the book of M. A. Krasnosel'skii and Ya. B. Rutickii [1]. The special analysis of integral operators of potential type were given by S. L. Sobolev [1]. Important conditions for the complete continuity of linear integral operators were found by L. V. Kantorovič [1]. Some new theorems were indicated by M. A. Krasnosel'skii and E. I. Pustyl'nik [1].

Conditions for the complete continuity of non-linear integral operators were established by many authors beginning with V. V. Nemyckii [1, 2]. The most general conditions for the complete continuity of the operator (7.6) in space C were found by L. A. Ladyženskii [1, 2]. Theorem 7.1 (in another formulation) was proved by M. A. Krasnosel'skii [5]. A further strengthening is given by M. A. Krasnosel'skii and E. I. Pustyl'nik [1]. Conditions for the complete continuity of non-linear operators in Orlicz spaces were indicated by M. A. Krasnosel'skii and Ya. B. Rutickii [1].

Conditions for the differentiability of integral operators were established by many authors; in this section there are no new results in this direction. Subsections 7.1.4 and 7.1.5 were written following the article of M. A. Krasnosel'skii and L. A. Ladyženskii [3].

As we have already mentioned, several theorems about positive characteristic functions of integral operators with concave non-linearities were established by P. S. Uryson [1]. The method of Uryson was applied and developed by a number of authors; we note in this connection the papers of I. A. Bakhtin [4], A. I. Guseinov [1], A. I. Guseinov and Ya. S.

Mamedov [1], Ya. S. Mamedov [1–5]. For some highly specialized families of non-linear integral operators, uniqueness theorems for positive characteristic functions were proved by the methods of the theory of cones by M. A. Rutman (see M. G. Kreĭn and M. A. Rutman [1]). Positive characteristic functions of non-linear integral operators were studied by M. A. Krasnosel'skii [4, 5], L. A. Ladyženskii [2], I. A. Bakhtin [4], and others. The uniqueness theorems cited in this section for a positive solution for the case of concave non-linearities actually are generalizations of the theorems of Uryson who employed another method of investigation. The uniqueness conditions for equations with convex non-linearities are indicated apparently for the first time.

By virtue of Theorems 6.6 and 6.7 the positive solutions of integral equations with concave non-linearities can be obtained as a limit of successive approximations. This fact (for some additional assumptions) is contained partially in an article of P. S. Uryson [1]; the constructions in the article are essentially based on the special properties of an initial approximation where a method of the construction of an initial approximation is not indicated having these properties and only the existence of such an approximation is proved. P. S. Uryson in [1] did not interest himself with the question about the convergence of successive approximations since in his constructions they play an important but auxiliary role.

7.2. General facts involving elliptic operators are indicated in the book of Miranda [1].

The integral inequality (7.68) and Theorem 7.9 were established by M. A. Krasnosel'skii and P. S. Sobolevskii [1]; in the article indicated the question is considered what smoothness properties the coefficients of the differential operators and boundary of the domain must have in order to satisfy an integral inequality. It is of interest to establish the validity of analogous inequalities for the case of domains with piecewise smooth boundaries. The remaining results were not previously noted as far as we know.

7.3. In the part involving the general theory of differential equations, only well known facts discussed by many authors are cited (see, for example, F. P. Gantmakher [1], G. Sansone [1]). The Poincaré principle of pointwise mapping obtained a wide development in the works of the Gor'kii mathematicians. The choice of invariant cones for the application of the Poincaré principle was indicated (apparently for the first time) by M. A. Krasnosel'skii [9].

7.4. The fundamental paper with respect to a two point problem is due to S. N. Bernstein [1]. This paper served as a starting point for many investigations; many results are summarized in the book of G. Sansone [1]. Methods of functional analysis were applied to a two point problem by M. A. Krasnosel'skii [3], A. L. Perov [1], M. P. Semyonov [1] and others. The problem about longitudinal bending of a compressed rod reduces to a two point boundary problem for some concrete equation of second order; this problem was investigated by M. A. Krasnosel'skii and I. A. Bakhtin (see M. A. Krasnosel'skii [4, 5], M. A. Krasnosel'skii and I. A. Bakhtin [1]) with the methods of functional analysis; part of the results was obtained later by Ya. D. Mamedov [5] by another method. We note that the investigation of I. A. Bakhtin and M. A. Krasnosel'skii of the first form of loss of elasticity of a compressed rod is based on the reduction of the problem to an operator equation with a concave operator on some cone.

The facts investigated in the section involving a linear two point problem were investigated by Sansone [1] in a new form.

The theorems about non-negative solutions of a two point boundary value problem for a non-linear scalar equation, proved in the section, can be obtained by methods based on the study of angular functions of solutions (see J. Sanson [1], A. L. Perov [1]). In particular, statements containing Theorems 7.22 and 7.23 are proved in the paper of A. I. Perov [1].

At the end of subsection 7.4.4 a remark was made about the uniqueness of solutions for a two point problem in the case when the function $f(x)$ is convex in equation (7.170). Apparently, this fact is known; however, we do not know where it is proved in general form. The proof of the indicated fact was given to us by A. I. Perov; its proof in regard to its methods is far from those of the present book.

The considerations investigated in subsections 7.4.5 and 7.4.6 belong to I. Ya. Bakel'man and M. A. Krasnosel'skii [2].

7.5. The general plan of the applications of the methods of functional analysis in the theory of non-linear oscillations, applied in this section, was indicated by M. A. Krasnosel'skii [6, 7].

7.6. The study of equations with Monge-Ampère operators was developed beginning with the classical papers of S. N. Bernstein by many mathematicians: A. D. Aleksandrov, A. V. Pogorelov, Leray and others. Topological methods (Schauder's principle) were applied, apparently for

the first time, in the theory of Monge-Ampère operator equations by I. Ya. Bakel'man [5]. In this connection, I. Ya. Bakel'man used, on the one hand, geometrical methods which were created and developed in the papers of A. D. Aleksandrov and A. V. Pogorelov and, on the other hand, proposed new functional analysis concepts.

The theorems investigated in this section were obtained jointly by I. Ya. Bakel'man and M. A. Krasnosel'skii [1].

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